Polynomials are one of principal tools of classical numerical analysis. When a function needs to be interpolated, integrated, differentiated, etc., it is assumed to be approximated by a polynomial of a certain fixed order (though the polynomial is almost never constructed explicitly), and a treatment appropriate to such a polynomial is applied. We introduce analogous techniques based on the assumption that the function to be dealt with is band-limited, and use the well-developed apparatus of Prolate Spheroidal Wave Functions to construct quadratures, interpolation and differentiation formulae, etc. for band-limited functions. Since band-limited functions are often encountered in physics, engineering, statistics, etc. the apparatus we introduce appears to be natural in many environments. Our results are illustrated with several numerical examples.

Prolate Spheroidal Wave Functions, Quadrature, and Interpolation

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Prolate Spheroidal Wave Functions, Quadrature, and Interpolation

1 Introduction

Numerical quadrature and interpolation are a well-developed part of numerical analysis; polynomials are the classical tool for the design of such schemes. Conceptually speaking, one assumes that the function is well-approximated by expressions of the form

$$\sum_{j=0}^{n} a_j x^j, \tag{1}$$

with reasonably small n, and designs algorithms that are effective for functions of the form (1) (needless to say, one almost never actually computes the coefficients $\{a_i\}$; one only uses the fact of their existence). Obviously, the polynomial approach is only effective for functions that are well-approximated by polynomials.

When one has to handle functions that are well-behaved on the whole line (for example, in signal processing), polynomials are not an appropriate tool. In such cases, trigonometric polynomials are used; existing tools are very satisfactory for dealing with functions defined and well-behaved on the whole of $\mathbb{R}^1$. Such tools, in effect, make the assumption that the functions are band-limited or nearly so; a function $f : \mathbb{R} \to \mathbb{R}$ is said to be band-limited if there exist a positive real c and a function $\sigma \in L^2[-1,1]$ such that

$$f(x) = \int_{-1}^{1} e^{icxt}\, \sigma(t)\; dt. \tag{2}$$

However, in many cases, we are confronted with band-limited functions defined on intervals (or, more generally, on compact regions in $\mathbb{R}^n$). Wave phenomena are a rich source of such functions, both in the engineering and computational contexts; they are also encountered in fluid dynamics, signal processing, and many other areas. Often, such functions can be effectively approximated by polynomials via standard tools of classical analysis. However, even when such approximations are feasible, they are usually not optimal. Smooth periodic functions are a good illustration of this observation: while they *can* be approximated by polynomials (for example, via Chebyshev or Legendre expansions), they are more efficiently approximated by Fourier expansions, both for analytical and numerical purposes. It would appear that an approach explicitly based on trigonometric polynomials could be more efficient in dealing with band-limited functions.

In the engineering context, such an apparatus was constructed more than 30 years ago (see [20]-[21], [7]-[9]). The natural tool for analyzing band-limited functions on $\mathbb{R}^1$ is the Fourier Transform, unless the functions are periodic, in which case the natural tool is the Fourier Series. The authors of [20]-[21] observe that for the analysis of band-limited functions on the interval, Prolate Spheroidal Wave Functions are likewise a natural approach. The authors also construct a multidimensional version of the theory, though their apparatus is only complete for the case of spherical regions.

The present paper constructs tools for the use of the approach of [20]-[21] in the modern computational environment. We construct a class of quadratures for band-limited functions that closely parallel the Gaussian quadratures for polynomials. The nodes are very close to being roots of appropriately chosen Prolate Spheroidal Wave Functions, the resulting quadratures are stable, and all weights are positive. As in the case of polynomials, there are interpolation, differentiation and indefinite integration schemes associated with the obtained quadratures, exact on certain classes of band-limited functions. These procedures are the main tools necessary for the numerical use of spectral discretizations based on Prolate Spheroidal Wave Functions, instead of on the usual polynomial bases. When dealing with band-limited functions, the number of nodes required by these procedures to obtain a prescribed accuracy is much less than that required by their polynomial-based counterparts. An additional bonus is the fact that the condition number of differentiation of prolate spheroidal wave functions is less than that of differentiation of the usual polynomial basis functions (see Section 8 below).

This paper is organized as follows. Section 2 summarizes various standard mathematical facts used in the remainder of the paper. Section 3 contains derivations of various results used in the algorithms described in later sections. Section 4 describes algorithms for evaluation of prolate spheroidal wave functions and associated eigenvalues. Section 5 describes a construction of quadratures for band-limited functions. Section 6 describes an alternative approach to arriving at such quadratures; it shows that roots of appropriately chosen prolate spheroidal wave functions can serve as quadrature nodes. Section 7 analyzes the use of prolate spheroidal wave functions for interpolation. Section 8 contains results of our numerical experiments with quadratures and interpolation. Section 9 contains a number of miscellaneous properties of prolate spheroidal wave functions, and Section 10 contains generalizations and conclusions.

2 Mathematical Preliminaries

As a matter of convention, in this paper the norm of a function is, unless stated otherwise, its L^2 norm:

$$||f|| = \sqrt{\int |f(x)|^2 \, dx}. \tag{3}$$

2.1 Chebyshev systems

Definition 2.1 *A sequence of functions $\phi_1, \ldots, \phi_n$ will be referred to as a* Chebyshev system *on the interval $[a, b]$ if each of them is continuous and the determinant*

$$\begin{vmatrix} \phi_1(x_1) & \cdots & \phi_1(x_n) \\ \vdots & & \vdots \\ \phi_n(x_1) & \cdots & \phi_n(x_n) \end{vmatrix} \tag{4}$$

is nonzero for any sequence of points $x_1, \ldots, x_n$ such that $a \leq x_1 < x_2 \ldots < x_n \leq b$.

An alternate definition of a Chebyshev system is that any linear combination of the functions with nonzero coefficients must have fewer than n zeros.

Examples of Chebyshev and extended Chebyshev systems include the following (additional examples can be found in [11]).

Example 2.1 *The powers $1, x, x^2, \ldots, x^n$ form an extended Chebyshev system on the interval $(-\infty, \infty)$.*

Example 2.2 *The exponentials $e^{-\lambda_1 x}, e^{-\lambda_2 x}, \ldots, e^{-\lambda_n x}$ form an extended Chebyshev system for any $\lambda_1, \ldots, \lambda_n > 0$ on the interval $[0, \infty)$.*

Example 2.3 *The functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx$ form a Chebyshev system on the interval $[0, 2\pi]$.*

2.2 Generalized Gaussian quadratures

A quadrature rule is an expression of the form

$$\sum_{j=1}^{n} w_j \phi(x_j), \tag{5}$$

where the points $x_j \in \mathbb{R}$ and coefficients $w_j \in \mathbb{R}$ are referred to as the nodes and weights of the quadrature, respectively. They serve as approximations to integrals of the form

$$\int_a^b \phi(x)\, \omega(x)\, dx, \tag{6}$$

with ω being an integrable non-negative function.

Quadratures are typically chosen so that the quadrature (5) is equal to the desired integral (6) for some set of functions, commonly polynomials of some fixed order. Of these, the classical Gaussian quadrature rules consist of n nodes and integrate polynomials of order $2n - 1$ exactly. In [13], the notion of a Gaussian quadrature was generalized as follows:

Definition 2.2 *A quadrature formula will be referred to as* Gaussian *with respect to a set of $2n$ functions $\phi_1, \ldots, \phi_{2n} : [a,b] \to \mathbb{R}$ and a weight function $\omega : [a,b] \to \mathbb{R}^+$, if it consists of n weights and nodes, and integrates the functions ϕ_i exactly with the weight function ω for all $i = 1, \ldots, 2n$. The weights and nodes of a Gaussian quadrature will be referred to as Gaussian weights and nodes respectively.*

The following theorem appears to be due to Markov [14, 15]; proofs of it can also be found in [12] and [11] (in a somewhat different form).

Theorem 2.1 *Suppose that the functions $\phi_1, \ldots, \phi_{2n} : [a,b] \to \mathbb{R}$ form a Chebyshev system on $[a,b]$. Suppose in addition that $\omega : [a,b] \to \mathbb{R}$ is a non-negative integrable function $[a,b] \to \mathbb{R}$. Then there exists a unique Gaussian quadrature for the functions $\phi_1, \ldots, \phi_{2n}$ on $[a,b]$ with respect to the weight function ω. The weights of this quadrature are positive.*

While the existence of Generalized Gaussian Quadratures was observed more than 100 years ago, the constructions found in [14, 15], [6, 12], [10, 11] do not easily yield numerical algorithms for the design of such quadrature formulae; such algorithms have been constructed recently (see [13, 25, 2]).

Remark 2.1 It might be worthwhile to observe here that when a Generalized Gaussian quadrature is to be constructed, the determination of its nodes tends to be the critical step (though the procedure of [13, 25, 2] determines the nodes and weights simultaneously). Indeed, once the nodes $x_1, x_2, \ldots, x_n$ have been found, the weights $w_1, w_2, \ldots, w_n$ can be determined easily as the solution of the $n \times n$ system of linear equations

$$\sum_{j=1}^{n} w_j \cdot \phi_i(x_j) = \int_a^b \phi_i(x)\, dx, \tag{7}$$

with $i = 1, 2, \ldots, n$.

2.3 Legendre Polynomials

In agreement with standard practice, we will be denoting by P_n the classical Legendre polynomials, defined by the three-term recursion

$$P_{n+1}(x) = \frac{2n+1}{n+1}\, x\, P_n(x) - \frac{n}{n+1}\, P_{n-1}(x), \tag{8}$$

with the initial conditions

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x; \end{aligned} \tag{9}$$

as is well-known,

$$P_k(1) = 1 \tag{10}$$

for all $k = 0, 1, 2, \ldots$, and each of the polynomials P_k satisfies the differential equation

$$(1 - x^2)\,\frac{d^2 P_k(x)}{dx^2} - 2x\,\frac{dP_k(x)}{dx} + k\cdot(k+1)\,P_k(x) = 0. \tag{11}$$

The polynomials defined by the formulae (8),(9) are orthogonal on the interval $[-1, 1]$; however, they are not orthonormal, since for each $n \geq 0$,

$$\int_{-1}^{1} (P_n(x))^2\, dx = \frac{1}{n + 1/2}; \tag{12}$$

the normalized version of the Legendre polynomials will be denoted by $\overline{P_n}$, so that

$$\overline{P_n}(x) = P_n(x) \cdot \sqrt{n + 1/2}. \tag{13}$$

The following lemma follows immediately from the Cauchy-Schwartz inequality and from the orthogonality of the Legendre polynomials on the interval $[-1, 1]$:

Lemma 2.2 *For all integer $k \geq n$,*

$$\left| \int_{-1}^{1} x^k\, \overline{P_n}(x)\, dx \right| < \sqrt{\frac{2}{k+1}}. \tag{14}$$

For all integer $0 \leq k < n$,

$$\left| \int_{-1}^{1} x^k\, \overline{P_n}(x)\, dx \right| = 0. \tag{15}$$

2.4 Convolutional Volterra Equations

A convolutional Volterra equation of the second kind is an expression of the form

$$\varphi(x) = \int_a^x K(x - t)\,\varphi(t)\, dt \; + \; \sigma(x) \tag{16}$$

where a, b are a pair of numbers such that $a < b$, the functions $\sigma, K : [a, b] \to \mathbb{C}$ are square-integrable, and $\varphi : [a, b] \to \mathbb{C}$ is the function to be determined. Proofs of the following theorem can be found in [4], as well as in many other sources.

Theorem 2.3 *The equation (16) always has a unique solution on the interval $[a, b]$. If both functions K, σ are k times continuously differentiable, the solution φ is also k times continuously differentiable.*

2.5 Prolate Spheroidal Wave Functions

In this subsection, we summarize certain facts about the Prolate Spheroidal Wave Functions. Unless stated otherwise, all these facts can be found in [20, 17].

Given a real $c > 0$, we will denote by F_c the operator $L^2[-1,1] \to L^2[-1,1]$ defined by the formula

$$F_c(\varphi)(x) = \int_{-1}^{1} e^{icxt}\,\varphi(t)\,dt. \tag{17}$$

Obviously, F_c is compact; we will denote by $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ the eigenvalues of F_c ordered so that $|\lambda_{j-1}| \geq |\lambda_j|$ for all natural j. For each non-negative integer j, we will denote by ψ_j the eigenfunctions corresponding to λ_j, so that

$$\lambda_j \psi_j(x) = \int_{-1}^{1} e^{icxt}\,\psi_j(t)\,dt, \tag{18}$$

for all $x \in [-1,1]$; we adopt the convention that the functions are normalized such that $||\psi_j||_{L^2[-1,1]} = 1$, for all j.[1] The following theorem is a combination of several lemmas from [20],[6],[11].

Theorem 2.4 *For any positive real c, the eigenfunctions $\psi_0, \psi_1, \ldots,$ of the operator F_c are purely real, are orthonormal, and are complete in $L^2[-1,1]$. The even-numbered eigenfunctions are even, and the odd-numbered ones are odd. All eigenvalues of F_c are non-zero and simple; the even-numbered eigenvalues are purely real, and the odd-numbered ones are purely imaginary; in particular, $\lambda_j = i^j|\lambda_j|$. The functions ψ_i constitute a Chebychev system on the interval $[-1,1]$; in particular, the function ψ_i has exactly i zeroes on that interval, for any $i = 0, 1, \ldots,$.*

We will define the self-adjoint operator $Q_c : L^2[-1,1] \to L^2[-1,1]$ by the formula

$$Q_c(\varphi) = \frac{1}{\pi}\int_{-1}^{1} \frac{sin(c \cdot (x-t))}{x-t}\,\varphi(t)\,dt; \tag{19}$$

a simple calculation shows that

$$Q_c = \frac{c}{2\pi} \cdot F_c^* \cdot F_c, \tag{20}$$

that Q_c has the same eigenfunctions as F_c, and that the j-th (in descending order) eigenvalue μ_j of Q_c is connected with λ_j by the formula

$$\mu_j = \frac{c}{2\pi} \cdot |\lambda_j|^2. \tag{21}$$

[1] This convention differs from that used in [20]; however, the present paper is concerned almost exclusively with approximation of functions on $[-1,1]$, and in that context, the convention that the functions $\{\psi_j\}$ have unit norm on that interval is by far the most convenient.

The operator Q_c is obviously closely related to the operator $P_c : L^2[-\infty, \infty] \to [-\infty, \infty]$ defined by the formula

$$P_c(\varphi) = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{sin(c \cdot (x-t))}{x-t} \cdot \varphi(t) dt, \tag{22}$$

which, as is well known, is the orthogonal projection operator onto the space of functions of band limit c on $(-\infty, \infty)$.

For large c, the spectrum of Q_c consists of three parts: about $2c/\pi$ eigenvalues that are very close to 1, followed by order $\log(c)$ eigenvalues which decay exponentially from 1 to nearly 0; the remaining eigenvalues are all very close to zero. The following theorem, proven (in a slightly different form) in [19], describes the spectrum of Q_c more precisely.

Theorem 2.5 *For any positive real c and $0 < \alpha < 1$ the number N of eigenvalues of the operator Q_c that are greater than α satisfies the inequality*

$$\begin{aligned} &\frac{2c}{\pi} + \left(\frac{1}{\pi^2} \log \frac{1-\alpha}{\alpha}\right) \log(c) - 10 \cdot \log(c) < N < \\ &\frac{2c}{\pi} + \left(\frac{1}{\pi^2} \log \frac{1-\alpha}{\alpha}\right) \log(c) + 10 \cdot \log(c). \end{aligned} \tag{23}$$

By a remarkable coincidence, the eigenfunctions $\psi_0, \psi_1, \cdots, \psi_n$ of the operator Q_c turn out to be the Prolate Spheroidal Wave functions, well-known from classical Mathematical Physics (see, for example, [16]). The following theorem formalizes this statement; it is proven in a considerably more general form in [21].

Theorem 2.6 *For any $c > 0$, there exists a strictly increasing sequence of positive real numbers $\chi_0, \chi_1, \ldots$ such that for each $j \geq 0$, the differential equation*

$$(1-x^2)\, \psi''(x) - 2x\, \psi'(x) + (\chi_j - c^2 x^2)\, \psi(x) = 0 \tag{24}$$

has a solution that is continuous on the interval $[-1, 1]$. For each $j \geq 0$, the function ψ_j (defined in Theorem 2.4) is the solution of (24).

3 Analytical Apparatus

3.1 Prolate Series

Since the functions $\psi_0, \psi_1, \ldots, \psi_n, \ldots$ are a complete orthonormal basis in $L^2[-1, 1]$, any formula for the inner product of prolate spheroidal wave functions with another function f is also a formula for the coefficients of an expansion of f into prolate spheroidal functions (which we will refer to as the prolate expansion of f). Thus the following theorem provides the coefficients of the prolate expansion of the derivative of a prolate spheroidal function, and also the coefficients of the prolate expansion of a prolate spheroidal wave function multiplied by x. Those coefficients are also the entries of the matrix for differentiation of a prolate expansion (producing another prolate expansion), and the entries of the matrix for multiplication of a prolate expansion by x, respectively. (These formulae are not, however, suitable for producing such matrices numerically, since in many cases they exhibit catastrophic cancellation.)

Theorem 3.1 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m = n \pmod 2$, then*

$$\int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\; dx = \int_{-1}^{1} x\, \psi_n(x)\, \psi_m(x)\; dx = 0. \tag{25}$$

If $m \neq n \pmod 2$, then

$$\int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\; dx = \frac{2\,\lambda_m^2}{\lambda_m^2 + \lambda_n^2}\, \psi_m(1)\, \psi_n(1)\,, \tag{26}$$

$$\int_{-1}^{1} x\, \psi_n(x)\, \psi_m(x)\; dx = \frac{2}{ic}\, \frac{\lambda_m \lambda_n}{\lambda_m^2 + \lambda_n^2}\, \psi_m(1)\, \psi_n(1)\,. \tag{27}$$

Proof. Since the functions ψ_j are alternately even and odd, (25) is obvious. In order to prove (26), we start with the identity

$$\lambda_n \psi_n = \int_{-1}^{1} e^{icxt}\, \psi_n(t)\; dt \tag{28}$$

(see (18) in Subsection 2.5). Differentiating (28) with respect to x, we obtain

$$\lambda_n \psi_n'(x) = i\,c \int_{-1}^{1} t\, e^{icxt}\, \psi_n(t)\; dt. \tag{29}$$

Projecting both sides of (29) on ψ_m and using the identity (28) (with n replaced with m) again, we have

$$\begin{aligned}
&\lambda_n \int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\; dx \\
&= i\,c \int_{-1}^{1} \psi_m(x) \int_{-1}^{1} t\, e^{icxt}\, \psi_n(t)\; dt\, dx \\
&= i\,c \int_{-1}^{1} t\, \psi_n(t) \int_{-1}^{1} e^{icxt}\, \psi_m(x)\; dx\, dt \\
&= i\,c\lambda_m \int_{-1}^{1} t\, \psi_n(t)\, \psi_m(t)\; dt.
\end{aligned} \tag{30}$$

Obviously, the above calculation can be repeated with m and n exchanged, yielding the identity

$$\lambda_m \int_{-1}^{1} \psi_m'(x)\,\psi_n(x)\;dx = i\,c\,\lambda_n \int_{-1}^{1} t\,\psi_n(t)\,\psi_m(t)\;dt\,; \tag{31}$$

combining (30) with (31), we have

$$\int_{-1}^{1} \psi_m'(x)\,\psi_n(x)\;dx = \frac{\lambda_n^2}{\lambda_m^2} \int_{-1}^{1} \psi_m(x)\,\psi_n'(x)\;dx. \tag{32}$$

On the other hand, integrating the left side of (32) by parts, we have

$$\begin{aligned} &\int_{-1}^{1} \psi_m'(x)\,\psi_n(x)\;dx \\ &= \;\; \psi_m(1)\,\psi_n(1) \;-\; \psi_m(-1)\,\psi_n(-1) \;-\; \int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx. \end{aligned} \tag{33}$$

Since $m \neq n \pmod 2$, we rewrite (33) as

$$\begin{aligned} &\int_{-1}^{1} \psi_m'(x)\,\psi_n(x)\;dx \\ &= \;\; 2\,\psi_m(1)\,\psi_n(1) - \int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx. \end{aligned} \tag{34}$$

Now, combining (32) and (34) and rearranging terms, we get

$$\int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx = \frac{2\,\lambda_m^2}{\lambda_m^2+\lambda_n^2}\,\psi_m(1)\,\psi_n(1)\,. \tag{35}$$

Substituting (30) into (35), we get

$$\begin{aligned} &\int_{-1}^{1} x\,\psi_n(x)\,\psi_m(x)\;dx \\ &= \;\; \frac{1}{ic}\,\frac{\lambda_n}{\lambda_m} \int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx \\ &= \;\; \frac{1}{ic}\,\frac{\lambda_n}{\lambda_m}\,\frac{2\,\lambda_m^2}{\lambda_m^2+\lambda_n^2}\,\psi_m(1)\,\psi_n(1) \\ &= \;\; \frac{2}{ic}\,\frac{\lambda_m\lambda_n}{\lambda_m^2+\lambda_n^2}\,\psi_m(1)\,\psi_n(1). \end{aligned} \tag{36}$$

□

The following corollary, which is an immediate consequence of (32), finds use in the numerical evaluation of the eigenvalues $\{\lambda_j\}$:

Corollary 3.2 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m \neq n \pmod 2$, then*

$$\frac{\lambda_m^2}{\lambda_n^2} = \frac{\int_{-1}^{1} \psi_n'(x)\,\psi_m(x)\;dx}{\int_{-1}^{1} \psi_m'(x)\,\psi_n(x)\;dx}. \tag{37}$$

3.2 Decay of Legendre Coefficients of Prolate Spheroidal Wave-functions

Since each of the functions ψ_j is analytic on $\mathbb{C}$, on the interval $[-1,1]$ it can be expanded in a Legendre series of the form

$$\psi_j(x) = \sum_{k=0}^{\infty} \beta_k \overline{P_k}(x), \tag{38}$$

with the coefficients β_k decaying superalgebraically; the following two theorems establish bounds for the decay rate.

Lemma 3.3 *Let $\overline{P_n}(x)$ be the n-th normalized Legendre polynomial (defined in (13)). Then for any real a,*

$$\int_{-1}^{1} e^{iax}\,\overline{P_n}(x)\,dx$$
$$= \sum_{k=k_0}^{\infty} \alpha_k \int_{-1}^{1} x^{2k}\,\overline{P_n}(x)\,dx + i \sum_{k=k_0}^{\infty} \beta_k \int_{-1}^{1} x^{2k+1}\,\overline{P_n}(x)\,dx. \tag{39}$$

where

$$\alpha_k = (-1)^k \frac{a^{2k}}{(2k)!}, \tag{40}$$
$$\beta_k = (-1)^k \frac{a^{2k+1}}{(2k+1)!}, \tag{41}$$
$$k_0 = \lfloor n/2 \rfloor. \tag{42}$$

Furthermore, for all integer $m \geq \lfloor e \cdot |a| \rfloor + 1$,

$$\left| \int_{-1}^{1} e^{iax}\,\overline{P_n}(x)\;dx - \sum_{k=k_0}^{m-1} \alpha_k \int_{-1}^{1} x^{2k}\,\overline{P_n}(x)\;dx \right.$$
$$\left. - \, i \sum_{k=k_0}^{m-1} \beta_k \int_{-1}^{1} x^{2k+1}\,\overline{P_n}(x)\;dx \right| < \left(\frac{1}{2}\right)^{2m}. \tag{43}$$

In particular, if

$$n \geq 2\left(\lfloor e \cdot |a| \rfloor + 1\right), \tag{44}$$

then

$$\left| \int_{-1}^{1} e^{iax}\, \overline{P_n}(x)\, dx \right| < \left(\frac{1}{2}\right)^{n-1}. \tag{45}$$

Proof. The formula (39) follows immediately from Lemma 2.2 and Taylor's expansion of e^{iax}. In order to prove (43), we assume that m is an integer such that

$$m \geq \lfloor e \cdot |a| \rfloor + 1. \tag{46}$$

Introducing the notation

$$R_m = \sum_{k=m}^{\infty} \alpha_k \int_{-1}^{1} x^{2k}\, \overline{P_n}(x)\, dx + i \sum_{k=m}^{\infty} \beta_k \int_{-1}^{1} x^{2k+1}\, \overline{P_n}(x)\, dx, \tag{47}$$

we immediately observe that, due to Lemma 2.2 and the triangle inequality,

$$\begin{aligned} |R_m| &\leq \sum_{k=2m}^{\infty} \left(\frac{|a|^k}{k\,!} \cdot \sqrt{\frac{2}{k+1}} \right) \\ &< \sum_{k=2m}^{\infty} \frac{|a|^k}{k\,!}. \end{aligned} \tag{48}$$

Since (46) implies that

$$\frac{|a|}{2m+k} < \frac{|a|}{2m} < \frac{1}{2e} < \frac{1}{2}, \tag{49}$$

for all integer $m, k > 0$, we rewrite (48) as

$$\begin{aligned} |R_m| &< \frac{|a|^{2m}}{(2m)\,!} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) \\ &< 2\,\frac{|a|^{2m}}{(2m)\,!}, \end{aligned} \tag{50}$$

and obtain (43) immediately using Stirling's formula. Finally, we obtain (45) by choosing

$$m = \lfloor e \cdot |a| \rfloor + 1. \tag{51}$$

□

Theorem 3.4 *Let $\psi_m(x)$ be the m-th prolate spheroidal function with band limit c, let $\overline{P_k}(x)$ be the k-th normalized Legendre polynomial (defined in (13)), and let λ_m be the eigenvalue which corresponds to $\psi_m(x)$ (as in Theorem 2.4). Then for all integer $m \geq 0$ and all real positive c, if*

$$k \geq 2\left(\lfloor e \cdot c \rfloor + 1\right), \tag{52}$$

then

$$\left| \int_{-1}^{1} \psi_m(x)\, \overline{P_k}(x)\ dx \right| < \frac{1}{\lambda_m} \cdot \left(\frac{1}{2}\right)^{k-1}. \tag{53}$$

Moreover, given any $\varepsilon > 0$, if

$$k \geq 2\left(\lfloor e \cdot c \rfloor + 1\right) + \log_2\left(\frac{1}{\varepsilon}\right) + \log_2\left(\frac{1}{\lambda_m}\right), \tag{54}$$

then

$$\left| \int_{-1}^{1} \psi_m(x)\, \overline{P_k}(x)\ dx \right| < \varepsilon. \tag{55}$$

Proof. Obviously

$$\begin{aligned} &\left| \int_{-1}^{1} \psi_m(x)\, \overline{P_k}(x)\ dx \right| \\ &= \frac{1}{|\lambda_m|} \cdot \left| \int_{-1}^{1} \psi_m(x) \left(\int_{-1}^{1} e^{icxt}\, \overline{P_k}(t)\ dt \right) dx \right| \\ &< \frac{1}{|\lambda_m|} \int_{-1}^{1} |\psi_m(x)| \cdot \left| \int_{-1}^{1} e^{icxt}\, \overline{P_k}(t)\ dt \right| dx. \end{aligned} \tag{56}$$

Introducing the notation

$$a = cx, \tag{57}$$

and remembering that

$$\int_{-1}^{1} |\psi_m(x)|\, dx = 1, \tag{58}$$

we observe that the combination of (56), (57), (58), and Lemma 3.3 implies that

$$\begin{aligned} &\left| \int_{-1}^{1} \psi_m(x)\, P_k(x)\ dx \right| \\ &< \frac{1}{|\lambda_m|} \cdot \left(\frac{1}{2}\right)^{k-1} \int_{-1}^{1} |\psi_m(x)|\, dx \\ &= \frac{1}{|\lambda_m|} \left(\frac{1}{2}\right)^{k-1}. \end{aligned} \tag{59}$$

Substituting (54) into (53), we immediately see (55). □

4 Numerical Evaluation of Prolate Spheroidal Wave-functions

Both the classical Bouwkamp algorithm (see, for example, [1]) for the evaluation of the functions ψ_j, and the algorithm presented in this paper for the same task, are based on the expression of those functions as a Legendre series of the form

$$\psi_j(x) = \sum_{k=0}^{\infty} \alpha_k P_k(x); \tag{60}$$

since the functions ψ_j are smooth, the coefficients α_k decay superalgebraically (with bounds for that decay being given in Theorem 3.4). Substituting (60) into (24), and using (8) and (11), we obtain the well-known three-term recursion

$$\begin{aligned} &\frac{(k+2)(k+1)}{(2k+3)(2k+5)} \cdot c^2 \cdot \alpha_{k+2} + \\ &\left(k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} \cdot c^2 - \chi_j\right) \cdot \alpha_k + \\ &\frac{k(k-1)}{(2k-3)(2k-1)} \cdot c^2 \cdot \alpha_{k-2} = 0. \end{aligned} \tag{61}$$

Combining (61) with (13), we obtain the three-term recursion

$$\begin{aligned} &\frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+5)(2k+1)}} \cdot c^2 \cdot \beta^j_{k+2} + \\ &\left(k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} \cdot c^2 - \chi_j\right) \cdot \beta^j_k + \\ &\frac{k(k-1)}{(2k-1)\sqrt{(2k-3)(2k+1)}} \cdot c^2 \cdot \beta^j_{k-2} = 0 \end{aligned} \tag{62}$$

for the coefficients $\beta^j_0, \beta^j_1, \ldots$ of the expansion

$$\psi_j(x) = \sum_{k=0}^{\infty} \beta^j_k \cdot \overline{P_k}(x); \tag{63}$$

for each $j = 0, 1, 2, \ldots$, we will denote by β^j the vector in l^2 defined by the formula

$$\beta^j = (\beta^j_0, \beta^j_1, \beta^j_2, \ldots). \tag{64}$$

The following theorem restates the recursion (62) in a slightly different form.

Theorem 4.1 *The coefficients χ_i are the eigenvalues and the vectors β^i are the corresponding eigenvectors of the operator $l^2 \to l^2$ represented by the symmetric matrix A given by the formulae*

$$A_{k,k} = k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)} \cdot c^2, \tag{65}$$

$$A_{k,k+2} = \frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+1)(2k+5)}} \cdot c^2, \tag{66}$$

$$A_{k+2,k} = \frac{(k+2)(k+1)}{(2k+3)\sqrt{(2k+1)(2k+5)}} \cdot c^2, \tag{67}$$

for all $k = 0, 1, 2, \ldots$, with the remainder of the entries of the matrix being zero.

In other words, the recursion (62) can be rewritten in the form

$$(A - \chi_j \cdot I)(\beta^j) = 0, \tag{68}$$

where A is separable into two symmetric tridiagonal matrices A_{even} and A_{odd}, the first consisting of the elements of A with even-numbered rows and columns and the second consisting of the elements of A with odd-numbered rows and columns. While these two matrices are infinite, and their entries do not decay much with increasing row or column number, the eigenvectors $\{\beta^j\}$ of interest (those corresponding to the first m prolate spheroidal functions) lie almost entirely in the leading rows and columns of the matrices (as shown by Theorem 3.4). Thus the evaluation of prolate spheroidal functions can be performed by the following procedure:

- 1. Generate the leading k rows and columns of A, where k is given by (54).
- 2. Split the generated portion of A into A_{even} and A_{odd}, and use a solver for the symmetric tridiagonal eigenproblem (such as that in LAPACK) to compute their eigenvectors $\{\beta^j\}$ and eigenvalues $\{\chi_j\}$.
- 3. Use the obtained values of the coefficients $\beta_0^j, \beta_1^j, \beta_2^j, \ldots$ in the expansion (63) to evaluate the function ψ_j at arbitrary points on the interval $[-1, 1]$.

Obviously steps 1 and 2 can be performed as a precomputation, for any given value of c. As a numerical diagonalization of a positive definite tridiagonal matrix with well-separated eigenvalues, this precomputation stage is numerically robust and efficient, requiring $O(c\,m)$ operations to construct the Legendre expansions of the form (64) for the first m prolate spheroidal functions; each subsequent evaluation of a prolate spheroidal function takes $O(c)$ operations.

4.1 Numerical Evaluation of Eigenvalues

Although the above algorithm for the evaluation of prolate spheroidal wave functions also produces the eigenvalues $\{\chi_j\}$ of the differential operator (24), it does not produce the eigenvalues $\{\lambda_j\}$ of the integral operator F_c (defined in (17)). Some of those eigenvalues can be computed using the formula

$$\lambda_j \psi_j(x) = \int_{-1}^{1} e^{icxt}\, \psi_j(t)\, dt, \tag{69}$$

evaluating the integral on the right hand side numerically; however, that evaluation obviously has a condition number of about $1/\lambda_j$, and is thus inappropriate for computing small λ_j. A well-conditioned procedure is as follows:

- 1. Use (69) to calculate λ_0, evaluating the right hand side numerically, and with $x = 0$ (so that $\psi_0(x)$ is not small).
- 2. Use the calculated λ_0, together with Corollary 3.2, to compute the absolute values $|\lambda_j|$, for $j = 1, 2, ..., m$, computing each $|\lambda_j|$ from $|\lambda_{j-1}|$ (and again, evaluating the required integrals numerically).
- 3. Use the fact that $\lambda_j = i^j |\lambda_j|$ (see Theorem 2.4) to finish the computation.

5 Quadratures for Band-Limited Functions

Since the prolate spheroidal wave functions $\psi_0, \psi_1, \ldots, \psi_n, \ldots$ constitute a complete orthonormal basis in $L^2[-1,1]$ (see Theorem 2.4),

$$e^{icxt} = \sum_{j=0}^{\infty} \left(\int_{-1}^{1} e^{icx\tau}\, \psi_j(\tau)\, d\tau \right) \psi_j(t), \tag{70}$$

for all $x, t \in [-1, 1]$; substituting (18) into (70) yields

$$e^{icxt} = \sum_{j=0}^{\infty} \lambda_j\, \psi_j(x)\, \psi_j(t), \tag{71}$$

Thus if a quadrature integrates exactly the first n eigenfunctions, that is, if

$$\sum_{k=1}^{m} w_k \psi_j(x_k) = \int_{-1}^{1} \psi_j(x)\, dx, \tag{72}$$

for all $j = 0, 1, \ldots, n-1$, then the error of the quadrature when applied to a function $f(x) = e^{icax}$, with $a \in [-1, 1]$, is given by

$$\begin{aligned}
&\sum_{k=1}^{m} w_k e^{icax_k} - \int_{-1}^{1} e^{icax}\, dx \\
&\quad = \sum_{k=1}^{m} w_k \left(\sum_{j=0}^{\infty} \lambda_j\, \psi_j(a)\, \psi_j(x_k) \right) - \int_{-1}^{1} \left(\sum_{j=0}^{\infty} \lambda_j\, \psi_j(a)\, \psi_j(x) \right)\, dx \\
&\quad = \sum_{k=1}^{m} w_k \left(\sum_{j=n}^{\infty} \lambda_j\, \psi_j(a)\, \psi_j(x_k) \right) - \int_{-1}^{1} \left(\sum_{j=n}^{\infty} \lambda_j\, \psi_j(a)\, \psi_j(x) \right)\, dx. \qquad (73)
\end{aligned}$$

Due to the orthonormality of the functions $\{\psi_j\}$,

$$\left\| \sum_{j=n}^{\infty} \lambda_j\, \psi_j(a)\, \psi_j(x) \right\| = \sqrt{\sum_{j=n}^{\infty} |\lambda_j|^2}. \qquad (74)$$

From (74), it is obvious that the error of integration (73) is of roughly the same magnitude as λ_n, provided that n is in the range where the eigenvalues $\{\lambda_j\}$ are decreasing exponentially (as is the case for quadratures of any useful accuracy; see Theorem 2.5) and provided in addition that the weights $\{w_k\}$ are not large.

Now, the existence of an $n/2$-point quadrature that is exact for the first n Prolate Spheriodal Wave functions follows from the combination of Theorems 2.1, 2.4; an algorithm for the numerical evaluation of nodes and weights of such quadratures can be found in [2]. An alternative procedure for the construction of quadrature formulae for band-limited functions (leading to slightly different nodes and weights) is described in the following section; a numerical comparison of the two can be found in Section 8 below.

Remark 5.1 The above text considers only the error of integration of a single exponential. For a band-limited function $g : [-1, 1] \to \mathbb{C}$ given by the formula

$$g(x) = \int_{-1}^{1} G(t)\, e^{icxt}\, dt, \qquad (75)$$

for some function $G : [-1, 1] \to \mathbb{C}$, the error is obviously bounded by the formula

$$\left| \sum_{k=1}^{m} w_k g(x_k) - \int_{-1}^{1} g(x)\, dx \right| \le \varepsilon \cdot \|G\|, \qquad (76)$$

where ε is the maximum error of integration (73) of a single exponential, for any $t \in [-1, 1]$. While $\|G\|$ might be much larger than $\|g\|_{[-1,1]}$ (as it is if, for instance, $g = \psi_{30 \cdot n}$), if the same equation (75) is used to extend g to the rest of the real line, then by Parseval's formula $\|G\| = \|g\|_{(-\infty,\infty)}$; that is to say, although the error of such a quadrature when applied to a band-limited function is not bounded proportional to the norm of that function on the interval of integration, it is bounded proportional to the norm of that function on the entire real line.

6 Quadrature Nodes from Roots of Prolate Functions

An alternative to the approach of the previous section is to use roots of appropriate prolate spheroidal wave functions as quadrature nodes, with the weights determined via the procedure described in Remark 2.1. The following theorems provide a basis for this; numerically (see Section 8) the resulting quadrature nodes tend to be inferior to those produced by the optimization scheme of [13, 25, 2]; however, they are useful as starting points for that scheme, or as somewhat less efficient nodes which can be computed much more quickly.

6.1 Euclid Division Algorithm for Band-Limited Functions

The following two theorems constitute a straightforward extension to band-limited functions of Euclid's division algorithm for polynomials. Their proofs are quite simple, and are provided here for completeness, since the author failed to find them in the literature.

Theorem 6.1 *Suppose that $\sigma, \varphi : [0,1] \to \mathbb{C}$ are a pair of c^2-functions such that*

$$\varphi(1) \neq 0, \tag{77}$$

c is a positive real number, and the functions f, p are defined by the formulae

$$f(x) = \int_0^1 \sigma(t)\, e^{2icxt}\, dt, \tag{78}$$

$$p(x) = \int_0^1 \varphi(t)\, e^{icxt}\, dt. \tag{79}$$

Then there exist two c^1-functions $\eta, \xi : [0,1] \to \mathbb{C}$ such that

$$f(x) = p(x)\, q(x) + r(x) \tag{80}$$

for all $x \in \mathbb{R}$, with the functions $q, r : [0,1] \to \mathbb{R}$ defined by the formulae

$$q(x) = \int_0^1 \eta(t)\, e^{icxt}\, dt, \tag{81}$$

$$r(x) = \int_0^1 \xi(t)\, e^{icxt}\, dt. \tag{82}$$

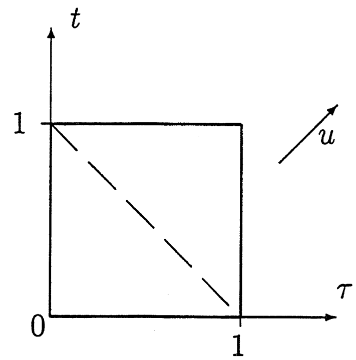


Figure 1: The split of integration range that yields (85)

Proof.

Obviously, for any functions p, q given by (79), (81),

$$p(x)\,q(x) = \int_0^1 \varphi(t)\,e^{icxt}\,dt \cdot \int_0^1 \eta(\tau)\,e^{icx\tau}\,d\tau$$
$$= \int_0^1 \int_0^1 \varphi(t)\,\eta(\tau)\,e^{icx(t+\tau)}\,d\tau\,dt. \tag{83}$$

Defining the new independent variable u by the formula

$$u = t + \tau, \tag{84}$$

we rewrite (83) as

$$p(x)q(x) = \int_0^1 e^{icux} \int_0^u \varphi(u-\tau)\,\eta(\tau)\,d\tau\,du$$
$$+ \int_1^2 e^{icux} \int_{u-1}^1 \varphi(u-\tau)\,\eta(\tau)\,d\tau\,du \tag{85}$$

(see Figure 1). Substituting (78), (82), and (85) into (80), we get

$$\int_0^1 e^{icux} \int_0^u \varphi(u-\tau)\,\eta(\tau)\,d\tau\,du$$
$$+ \int_1^2 e^{icux} \int_{u-1}^1 \varphi(u-\tau)\,\eta(\tau)\,d\tau\,du + \int_0^1 \xi(t)\,e^{icxt}\,dt$$
$$= \int_0^{1/2} \sigma(t)\,e^{2icxt}\,dt + \int_{1/2}^1 \sigma(t)\,e^{2icxt}\,dt. \tag{86}$$

Due to the well known uniqueness of the Fourier Transform, (86) is equivalent to two independent equations:

$$\int_0^1 e^{icux} \int_0^u \varphi(u-\tau)\,\eta(\tau)\,d\tau\,du + \int_0^1 \xi(t)\,e^{icxt}\,dt$$
$$= \int_0^{1/2} \sigma(t)\,e^{2icxt}\,dt, \tag{87}$$

$$\int_1^2 e^{icux} \int_{u-1}^1 \varphi(u-\tau)\, \eta(\tau)\, d\tau\, du = \int_{1/2}^1 \sigma(t)\, e^{2icxt}\, dt. \tag{88}$$

Now, we observe that (88) does not contain ξ, and use it to obtain an expression for η as a function of φ, σ. After that, we will view (87) as an expression for ξ via φ, σ, η.

From (88) and the uniqueness of the Fourier Transform, we obtain

$$\int_{u-1}^1 \varphi(u-\tau)\, \eta(\tau)\, d\tau = \sigma(\frac{u}{2}), \tag{89}$$

for all $u \in [1,2]$. Introducing the new variable v via the formula

$$v = u - 1, \tag{90}$$

we convert (89) into

$$\int_v^1 \varphi(v+1-\tau)\, \eta(\tau)\, d\tau = \sigma(\frac{v+1}{2}), \tag{91}$$

which is a Volterra equation of the first kind with respect to η; differentiating (91) with respect to v, we get

$$-\varphi(1)\, \eta(v) + \int_v^1 \varphi'(v+1-\tau)\, \eta(\tau)\, d\tau = \frac{1}{2}\sigma'(\frac{v+1}{2}), \tag{92}$$

which is a Volterra equation of the second kind. Now, the existence and uniqueness of the solution of (92) (and, therefore, of (89) and (88)) follows from Theorem 2.3 of Section 2.

With η defined as the solution of (89), we use (87) together with the uniqueness of the Fourier Transform, to finally obtain

$$\xi(u) = \sigma(\frac{u}{2}) - \int_0^u \varphi(u-\tau)\, \eta(\tau)\, d\tau, \tag{93}$$

for all $u \in [0,1]$.

□

The following theorem is a consequence of the preceding one.

Theorem 6.2 *Suppose that $\sigma, \varphi : [-1,1] \to \mathbb{C}$ are a pair of c^2−functions such that $\varphi(-1) \neq 0$, $\varphi(1) \neq 0$, c is a positive real number, and the functions f, p are defined by the formulae*

$$f(x) = \int_{-1}^1 \sigma(t)\, e^{2icxt}\, dt, \tag{94}$$

$$p(x) = \int_{-1}^1 \varphi(t)\, e^{icxt}\, dt. \tag{95}$$

Then there exist two c^1-functions $\eta, \xi : [-1,1] \to \mathbb{C}$ such that

$$f(x) = p(x)\, q(x) + r(x) \tag{96}$$

for all $x \in \mathbb{R}$, with the functions $q, r : [-1,1] \to \mathbb{R}$ defined by the formulae

$$q(x) = \int_{-1}^{1} \eta(t)\, e^{icxt}\, dt, \tag{97}$$

$$r(x) = \int_{-1}^{1} \xi(t)\, e^{icxt}\, dt. \tag{98}$$

Proof.

Defining the functions f_+, f_-,p_+, p_-, by the formulae

$$f_+(x) = \int_0^1 \sigma(t)\, e^{2icxt}\, dt, \tag{99}$$

$$f_-(x) = \int_{-1}^{0} \sigma(t)\, e^{2icxt}\, dt, \tag{100}$$

$$p_+(x) = \int_0^1 \varphi(t)\, e^{icxt}\, dt, \tag{101}$$

$$p_-(x) = \int_{-1}^{0} \varphi(t)\, e^{icxt}\, dt, \tag{102}$$

we observe that for all $x \in \mathbb{R}^1$,

$$f(x) = f_+(x) + f_-(x), \tag{103}$$

$$p(x) = p_+(x) + p_-(x). \tag{104}$$

Due to Theorem 6.1, there exist such η_+, η_-, ξ_+, ξ_-, that

$$f_+(x) = p_+(x)\, q_+(x) + r_+(x), \tag{105}$$

$$f_-(x) = p_-(x)\, q_-(x) + r_-(x), \tag{106}$$

with the functions q_+, q_-, r_+, r_- defined by the formulae

$$q_+(x) = \int_0^1 \eta_+(t)\, e^{icxt}\, dt, \tag{107}$$

$$q_-(x) = \int_{-1}^{0} \eta_-(t)\, e^{icxt}\, dt, \tag{108}$$

$$r_+(x) = \int_0^1 \xi_+(t)\, e^{icxt}\, dt, \tag{109}$$

$$r_-(x) = \int_{-1}^0 \xi_-(t)\, e^{icxt}\, dt. \tag{110}$$

Now, defining q, by the formula

$$q(x) = q_-(x) + q_+(x) \tag{111}$$

for all $x \in [-1, 1]$, we have

$$\begin{aligned} p(x)\, q(x) &= (p_-(x) + p_+(x)) \cdot (q_-(x) + q_+(x)) \\ &= p_+(x)\, q_+(x) + p_-(x)\, q_-(x) + p_-(x)\, q_+(x) + p_+(x)\, q_-(x), \end{aligned} \tag{112}$$

and we define $r(x)$ by the obvious formula

$$r(x) = r_-(x) + r_+(x) - (p_-(x)\, q_+(x) + p_+(x)\, q_-(x)). \tag{113}$$

□

6.2 Quadrature nodes from the division theorem

In much the same way that the division theorem for polynomials can be used to provide a constructive proof of Gaussian quadratures, Theorem 6.2 provides a method of constructing generalized Gaussian quadratures for band-limited functions. The method is as follows.

To construct a quadrature for functions of a bandwidth $2c$, prolate spheroidal wave functions corresponding to bandwidth c are used. (Thus the eigenvalues $\{\lambda_j\}$ and eigenfunctions $\{\psi_j\}$ are in this section, as elsewhere in the paper, those corresponding to bandwidth c). The following theorem provides a bound of the error of a quadrature whose nodes are the roots of the n'th prolate function ψ_n, when applied to a function f which satisfies the conditions of the division theorem, in terms of the norms of the quotient and remainder of f divided by ψ_n:

Theorem 6.3 *Suppose that $x_1, x_2, \ldots, x_n \in \mathbb{R}$ are the roots of ψ_n. Let the numbers $w_1, w_2, \ldots, w_n \in \mathbb{R}$ be such that*

$$\sum_{k=1}^{n} w_k \psi_j(x_k) = \int_{-1}^{1} \psi_j(x)\, dx, \tag{114}$$

for all $j = 0, 1, \ldots, n-1$. Then for any function $f : [-1,1] \to \mathbb{C}$ which satisfies the conditions of Theorem 6.2,

$$\left|\sum_{k=1}^{n} w_k f(x_k) - \int_{-1}^{1} f(x)\, dx\right| \leq |\lambda_n| \cdot ||\eta|| \; + \; ||\xi|| \cdot \sum_{j=n}^{\infty} |\lambda_j| \cdot ||\psi_j||_\infty^2 \cdot \left(2 + \sum_{k=1}^{m} ||w_k||\right), \tag{115}$$

where the functions $\eta, \xi : [-1,1] \to \mathbb{C}$ are as defined in Theorem 6.2.

Proof. Since f satisfies the conditions of Theorem 6.2, there exist functions $q, r : [-1,1] \to \mathbb{R}$ defined by (97),(98) such that

$$f(x) = \psi_n(x)\, q(x) + r(x). \tag{116}$$

Then, defining the error of integration E_f for the function f by

$$E_f = \left|\sum_{k=1}^{n} w_k f(x_k) - \int_{-1}^{1} f(x)\, dx\right| \tag{117}$$

we have

$$\begin{aligned} E_f &= \left|\sum_{k=1}^{n} w_k \left(\psi_n(x_k)\, q(x_k) + r(x_k)\right) - \int_{-1}^{1} \left(\psi_n(x)\, q(x) + r(x)\right) dx\right| \\ &\leq \left|\sum_{k=1}^{n} w_k\, \psi_n(x_k)\, q(x_k) - \int_{-1}^{1} \psi_n(x)\, q(x)\, dx\right| \\ &\quad + \left|\sum_{k=1}^{n} w_k r(x_k) - \int_{-1}^{1} r(x)\, dx\right| \end{aligned} \tag{118}$$

Since the nodes $\{x_k\}$ are the roots of ψ_n,

$$\sum_{k=1}^{n} w_k\, \psi_n(x_k)\, q(x_k) = 0. \tag{119}$$

Thus

$$E_f \leq \left|\int_{-1}^{1} \psi_n(x)\, q(x)\, dx\right| + \left|\sum_{k=1}^{n} w_k r(x_k) - \int_{-1}^{1} r(x)\, dx\right|. \tag{120}$$

Now

$$\begin{aligned} \int_{-1}^{1} \psi_n(x)\, q(x) dx &= \int_{-1}^{1} \psi_n(x) \int_{-1}^{1} \eta(t)\, e^{icxt}\, dt\, dx \\ &= \int_{-1}^{1} \eta(t) \int_{-1}^{1} \psi_n(x)\, e^{icxt}\, dx\, dt \\ &= \int_{-1}^{1} \eta(t)\, \lambda_n\, \psi_n(t)\, dt. \end{aligned} \tag{121}$$

Using the Cauchy-Schwartz inequality and the fact that the function ψ_n has unit norm, we get from (121) that

$$\left| \int_{-1}^{1} \psi_n(x)\, q(x)\, dx \right| \;\leq\; |\lambda_n| \cdot \|\eta\|. \tag{122}$$

Also,

$$\begin{aligned}
&\sum_{k=1}^{n} w_k r(x_k) - \int_{-1}^{1} r(x)\, dx \\
&\quad = \sum_{k=1}^{n} w_k \left(\int_{-1}^{1} \xi(t)\, e^{icx_k t}\, dt \right) - \int_{-1}^{1} \left(\int_{-1}^{1} \xi(t)\, e^{icxt}\, dt \right) dx \\
&\quad = \int_{-1}^{1} \xi(t) \left(\sum_{k=1}^{n} w_k e^{icx_k t} - \int_{-1}^{1} e^{icxt}\, dx \right) dt.
\end{aligned} \tag{123}$$

Substituting (73) into (123), and using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&\sum_{k=1}^{n} w_k r(x_k) - \int_{-1}^{1} r(x)\, dx \\
&\quad = \int_{-1}^{1} \xi(t) \left(\sum_{k=1}^{m} w_k \left(\sum_{j=n}^{\infty} \lambda_j\, \psi_j(t)\, \psi_j(x_k) \right) \right. \\
&\qquad\qquad \left. - \int_{-1}^{1} \left(\sum_{j=n}^{\infty} \lambda_j\, \psi_j(t)\, \psi_j(x) \right) dx \right) dt \\
&\quad \leq \|\xi\| \cdot \sum_{j=n}^{\infty} |\lambda_j| \cdot \|\psi_j\|_{\infty}^{2} \cdot \left(2 + \sum_{k=1}^{m} \|w_k\| \right).
\end{aligned} \tag{124}$$

Combining (120), (122), and (124), we get

$$E_f \;\leq\; |\lambda_n| \cdot \|\eta\| \;+\; \|\xi\| \cdot \sum_{j=n}^{\infty} |\lambda_j| \cdot \|\psi_j\|_{\infty}^{2} \cdot \left(2 + \sum_{k=1}^{m} \|w_k\| \right). \tag{125}$$

□

Remark 6.1 The use of Theorem 6.3 for the construction of quadrature rules for band-limited functions depends on the fact that the norms of the band-limited functions q and r in (116) are not large, compared to the norm of f (both sets of norms being on $[-\infty, \infty]$). Such estimates have been obtained for all $n > 2c/\pi + 10\log(c)$. The proofs are quite involved, and will be reported at a later date. In this paper, we demonstrate the performance of the obtained quadrature formulae numerically (see Section 8 below).

Remark 6.2 It is natural to view (116) as an analogue for band-limited functions of the Euclid division theorem for polynomials. However, there are certain differences. In particular, Theorem 6.1 admits extensions to band-limited functions of several variables, while the classical Euclid algorithm does not. Such extensions (together with several applications) will be reported at a later date.

7 Interpolation via Prolate Spheroidal Wavefunctions

Interpolation is usually performed by the following general procedure: assuming that the function $f : [a, b] \to \mathbb{C}$ to be interpolated is given by the formula

$$f(x) = c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x), \tag{126}$$

where $\phi_1, \phi_2, \ldots, \phi_n : [a, b] \to \mathbb{C}$ are a fixed sequence of functions (often polynomials), solve an $n \times n$ linear system to determine the coefficients $c_1, c_2, \ldots, c_n$ from the values of f at the n interpolation nodes, then use (126) to evaluate f wherever needed. As is well known, if f is well-approximated by a linear combination of the interpolation functions, and if the linear system to be solved is well-conditioned, then this procedure is accurate.

As shown in Section 5 in the context of quadratures, a linear combination of the first n prolate spheroidal functions $\psi_0, \psi_1, \ldots, \psi_{n-1}$ for a band limit c can provide a good approximation to functions of the form e^{icxt}, with $t \in [-1, 1]$ (see (71,74)); in the regime where the accuracy is numerically useful, the error is of the same order of magnitude as $|\lambda_n|$. This, in turn, shows that they provide a good approximation (in the same sense as in Remark 5.1) to any band-limited function of band limit c. Thus, if $\psi_0, \psi_1, \ldots, \psi_{n-1}$ are used as the interpolation functions in this procedure, they can be expected to yield an accurate interpolation scheme for band-limited functions, provided that the matrix to be inverted is well-conditioned. The following theorem shows that if the interpolation nodes are chosen to be quadrature nodes accurate up to twice the bandwidth of interpolation, with the quadrature formula being accurate to more than twice as many digits as the interpolation formula is to be accurate to, then the matrix inverted in the procedure is close to being a scaled version of an orthogonal matrix.

Theorem 7.1 *Suppose the numbers $w_1, w_2, \ldots, w_n \in \mathbb{R}$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$ are such that*

$$\left| \int_{-1}^{1} e^{2icax}\, dx - \sum_{j=1}^{n} w_j e^{2icax_j} \right| < \varepsilon, \tag{127}$$

for all $a \in [-1,1]$, and for some $c > 0$. Let the matrix A be given by the formula

$$A = \begin{pmatrix} \psi_0(x_1) & \psi_1(x_1) & \dots & \psi_{n-1}(x_1) \\ \psi_0(x_2) & \psi_1(x_2) & \dots & \psi_{n-1}(x_2) \\ \vdots & \vdots & & \vdots \\ \psi_0(x_n) & \psi_1(x_n) & \dots & \psi_{n-1}(x_n) \end{pmatrix}, \tag{128}$$

let the matrix W be the diagonal matrix whose diagonal entries are $w_1, w_2, \dots, w_n$, and let the matrix $E = [e_{jk}]$ be given by the formula

$$E = I - A^* W A. \tag{129}$$

Then

$$|e_{jk}| < \left| \frac{2\varepsilon}{\lambda_{j-1}\lambda_{k-1}} \right|. \tag{130}$$

Proof. Clearly

$$e_{jk} = \delta_{jk} - \sum_{l=1}^{n} w_l \, \psi_{j-1}(x_l) \, \psi_{k-1}(x_l), \tag{131}$$

where δ_{ij} is the Kronecker delta function. Using (18), this becomes

$$\begin{aligned} e_{jk} &= \delta_{jk} - \sum_{l=1}^{n} w_l \cdot \left(\frac{1}{\overline{\lambda}_{j-1}} \int_{-1}^{1} e^{-icx_l t} \, \psi_{j-1}(t) \, dt \right) \\ &\qquad \cdot \left(\frac{1}{\lambda_{k-1}} \int_{-1}^{1} e^{icx_l \tau} \, \psi_{k-1}(\tau) \, d\tau \right) \\ &= \delta_{jk} - \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1} \int_{-1}^{1} \psi_{j-1}(t) \, \psi_{k-1}(\tau) \sum_{l=1}^{n} w_l e^{-icx_l t} \, e^{icx_l \tau} \, dt \, d\tau. \end{aligned} \tag{132}$$

Using (127), this becomes

$$\begin{aligned} e_{jk} = \delta_{jk} - \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1} \int_{-1}^{1} & \psi_{j-1}(t) \, \psi_{k-1}(\tau) \\ & \cdot \left(\int_{-1}^{1} e^{-icst} \, e^{ics\tau} \, ds \; - \; f_\varepsilon(t+\tau) \right) \, dt \, d\tau, \end{aligned} \tag{133}$$

where $f_\varepsilon : [-2,2] \to \mathbb{C}$ is a function which satisfies the relation

$$|f_\varepsilon(x)| < \varepsilon, \tag{134}$$

for all $x \in [-2,2]$. Thus

$$\begin{aligned} e_{jk} &= \delta_{jk} - \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1} \int_{-1}^{1} \psi_{j-1}(t) \, \psi_{k-1}(\tau) \int_{-1}^{1} e^{-icst} \, e^{ics\tau} \, ds \, dt \, d\tau \\ &\quad + \frac{1}{\overline{\lambda}_{j-1}\lambda_{k-1}} \int_{-1}^{1} \int_{-1}^{1} \psi_{j-1}(t) \, \psi_{k-1}(\tau) \, f_\varepsilon(t+\tau) \, dt \, d\tau \end{aligned} \tag{135}$$

Using (18), this becomes

$$\begin{aligned} e_{jk} &= \delta_{jk} - \int_{-1}^{1} \psi_{j-1}(s)\,\psi_{k-1}(s)\;ds \\ &+ \frac{1}{\overline{\lambda_{j-1}\lambda_{k-1}}} \int_{-1}^{1} \psi_{k-1}(\tau) \int_{-1}^{1} \psi_{j-1}(t)\, f_\varepsilon(t+\tau)\;dt\,d\tau. \end{aligned} \tag{136}$$

Due to the orthonormality of the functions $\{\psi_j\}$, this becomes

$$e_{jk} = \frac{1}{\overline{\lambda_{j-1}\lambda_{k-1}}} \int_{-1}^{1} \psi_{k-1}(\tau) \int_{-1}^{1} \psi_{j-1}(t)\, f_\varepsilon(t+\tau)\;dt\,d\tau. \tag{137}$$

Using the Cauchy-Schwartz inequality, this becomes

$$\begin{aligned} |e_{jk}| &\leq \left|\frac{1}{\lambda_{j-1}\lambda_{k-1}}\right| \|\psi_{k-1}\| \sqrt{\int_{-1}^{1} \left|\int_{-1}^{1} \psi_{j-1}(t)\, f_\varepsilon(t+\tau)\;dt\right|^2 \;d\tau} \\ &\leq \left|\frac{1}{\lambda_{j-1}\lambda_{k-1}}\right| \sqrt{\int_{-1}^{1} \|\psi_{j-1}\|^2 \int_{-1}^{1} |f_\varepsilon(t+\tau)|^2\;dt\;d\tau} \\ &= \left|\frac{1}{\lambda_{j-1}\lambda_{k-1}}\right| \sqrt{\int_{-1}^{1}\int_{-1}^{1} |f_\varepsilon(t+\tau)|^2\;dt\;d\tau} \\ &< \left|\frac{2\varepsilon}{\lambda_{j-1}\lambda_{k-1}}\right|. \end{aligned} \tag{138}$$

□

From inspection of Theorem 2.5, it can easily be seen that the number N of eigenvalues needed for a bandwidth of $2c$ and an accuracy of ε^2 is roughly twice the number of eigenvalues needed for a bandwidth of c and an accuracy of ε. Thus a generalized Gaussian quadrature for a bandwidth $2c$ and an accuracy ε^2 has roughly the same number of nodes as are needed for interpolation of accuracy ε. In our numerical experiments, this correspondence was found to be much closer than the rough bounds in Theorem 2.5 indicate; in the results tabulated in Section 8, the number of nodes for an interpolation formula of a desired accuracy ε was always chosen to be the number of quadrature nodes for a desired accuracy ε^2 for twice the band limit (that number, in turn, being chosen as indicated in Section 5); the correspondence between the desired accuracy and the experimentally measured maximum error can be seen in Tables 3 and 4.

The coefficients $c_1, c_2, \ldots, c_n$ produced by this interpolation procedure (see (126)) can, of course, just as easily be used for evaluating derivatives or indefinite integrals of the interpolated function, as they can for computing the function itself.

8 Numerical Results

The algorithms of Sections 5–7 have been implemented in double precision (64-bit floating point) arithmetic, with results shown in Tables 1–4. Tables 1 and 2 show the performance of quadrature nodes produced by the schemes of Sections 5 and 6, when used as quadrature nodes; Tables 3 and 4 show their performance when used as interpolation nodes. These are not actually the same sets of nodes; even with the bandwidth c for interpolation being half of the bandwidth for quadrature (as it is in the tables), more nodes are needed to achieve a given accuracy of interpolation than are needed to achieve a given accuracy of quadrature, as can be seen by comparing the number of nodes (printed in the column labeled n in each table). The error figures in the tables are approximations of the maximum error of interpolation or of the quadrature, when applied to functions of the form $\cos(ax)$ and $\sin(ax)$, with $0 \leq a \leq c$; they were computed by measuring the error at a large number of points in a (for interpolation, in both a and x), including the extremes. The column labeled "Roots" contains the errors for the nodes produced by the scheme of Section 6; the column labeled "Refined" contains the errors after those nodes, used as a starting point, have been run through the scheme of Section 5. The variable ε which appears in the tables is the requested accuracy, used to determine the number of nodes in the ways described in Sections 5 and 7.

Also tabulated are the numbers of Legendre nodes required to achieve the same accuracy ε using polynomial interpolation or quadrature schemes. Since Chebyshev nodes are generally known to be superior for interpolation, for that case the numbers of Chebyshev nodes required to achieve the same accuracy are also tabulated.

Figure 2 contains the maximum norm of the derivative of each prolate function $\psi_j(x)$, for $c = 200$ and $x \in [-1, 1]$, as a function of j; also graphed, for comparison, is the maximum norm of the derivative of each normalized Legendre polynomial $\overline{P}_j(x)$ over the same range; and graphed below, on the same horizontal scale, are the norms of the eigenvalues λ_j. The graph shows that, for this value of c, computing the derivatives of a function given by a prolate series is a better-conditioned operation than computing the derivatives of a function given by a Legendre series of the same number of terms. (Obviously, if the number of terms can also be reduced, as in the situations of Tables 1–4, there is a further improvement in the condition number.) The same general pattern of behavior is exhibited for other values of c; as c approaches zero (and the prolate functions approach the Legendre polynomials), the value of j at which the maximum norm of the derivative rises sharply also approaches zero (as is to be expected, since for $c = 0$ the prolate functions reduce to Legendre polynomials). Finally, Tables 5 and 6 contain samples of quadrature weights and nodes.

Remark 8.1 In this paper, detailed discussion of issues encountered in the implementation of numerical algorithms has been deliberately avoided, as well as any discussion of

CPU time requirements, memory requirements, etc. Thus, we limit ourselves to observing that all algorithms have been implemented in FORTRAN, that with the exception of the procedure for the evaluation of Prolate Spheroidal Wave functions described in Section 4, we have not designed or implemented any new or original numerical algorithms, and that the procedure of Section 4 consists of applying standard tools of numerical analysis (diagonalization of a tridiagonal matrix) to the well-known recursion (61). The resulting algorithm for the evaluation of prolate spheroidal wave functions has the CPU time requirements proportional to c^2, with a fairly large proportionality constant. The procedure of [2], when applied to the system of functions $\psi_0, \psi_1, \ldots, \psi_{2n+1}$ requires order n^3 operations, also with a fairly large proportionality constant. On the other hand, the cost of finding all roots n of the function ψ_n lying on the interval $[-1, 1]$ is proportional to n, and the proportionality constant is not large. The largest c we have dealt with in our experiments was about 6000, with resulting quadratures having about 1900 nodes. In this regime, the construction of the quadrature (both nodes and weights) took several Mhz minutes on the 300-megaflop SUN workstation; while there are fairly obvious ways to reduce the cost of the calculation (both in terms of asymptotic CPU time requirements and in terms of associated proportionality constants) we have made no effort to do so.

The following observations can be made from the examples presented in this section, and from the more extensive tests performed by the authors.

1. When the nodes obtained via the algorithm of [2] are used for the integration of band-limited functions, the resulting quadrature rules are significantly more accurate than the quadratures obtained from the nodes of appropriately chosen prolate functions; however, the *difference* between the numbers of nodes required by the two approaches to obtain a *prescribed* precision is not large. When the nodes obtained via the two approaches are used for the interpolation (as opposed to the integration) of band-limited functions, the performances of the two are virtually identical.

2. For large c, the number of nodes required by a quadrature rule for the integration of band-limited functions with the band-limit c is close to $\frac{c}{\pi}$; the dependence on the required precision of integration is weak (as one would expect from Theorem 2.5 and subsequent developments).

3. The numbers of nodes required by our quadratures rules to integrate band-limited functions is roughly $\pi/2$ times less than the numbers of Gaussian nodes; the numbers of nodes required by our interpolation formulae in order to interpolate band-limited functions is roughly $\pi/2$ times less than the number of Chebychev (or Gaussian) nodes. Again, the dependence of the required number of nodes on the accuracy requirements is weak.

4. The norm of the differentiation operator based on our nodes is of the order $c^{3/2}$, as

compared to the norm of the spectral differentiation operators obtained from classical polynomial expansions; this might be useful in the design of spectral (or pseudospectral) techniques.

9 Miscellaneous Properties

Prolate spheroidal wave functions possess a rich set of properties, vaguely resembling the properties of Bessel functions. This section establishes some of those properties. Some of the identities below can be found in [20],[17],[5]; others are easily derivable from the former.

The identity

$$e^{icxt} = \sum_{j=0}^{\infty} \lambda_j \, \psi_j(x) \, \psi_j(t), \tag{139}$$

(see Section 5) has a number of consequences which, while fairly obvious, seem worth recording, since similar properties of other special functions have often been found useful. Differentiating (139) m times with respect to x and n times with respect to t yields the formula

$$x^m t^n e^{icxt} = \left(\frac{1}{ic}\right)^{(m+n)} \sum_{j=0}^{\infty} \lambda_j \, \psi_j^{(m)}(x) \, \psi_j^{(n)}(t), \tag{140}$$

for all $x, t \in [-1, 1]$. Multiplying (139) by e^{-icut}, and integrating with respect to t, converts it into

$$\frac{sin(c \cdot (x-u))}{x-u} = \frac{c}{2} \sum_{j=0}^{\infty} \lambda_j^2 \, \psi_j(x) \, \psi_j(u), \tag{141}$$

Taking the squared norm of (139), and integrating with respect to x and t, yields the formula

$$\sum_{j=0}^{\infty} |\lambda_j|^2 = 4; \tag{142}$$

combining this with (21) yields

$$\sum_{j=0}^{\infty} \mu_j = \frac{2c}{\pi}. \tag{143}$$

Setting $x = t = 1$ converts (139) into

$$e^{ic} = \sum_{j=0}^{\infty} \lambda_j \, \psi_j^2(1). \tag{144}$$

The identity

$$\lambda_j \psi_j(x) = \int_{-1}^{1} e^{icxt} \, \psi_j(t) \, dt \tag{145}$$

(see Section 2.5) also has a number of simple but potentially useful consequences. Differentiating it k times with respect to x, we get

$$\lambda_j \psi_j^{(k)}(x) = (ic)^k \int_{-1}^{1} e^{icxt} \, t^k \, \psi_j(t) \, dt. \tag{146}$$

We next consider the integral

$$f(x) = f(a,x) = \int_{-1}^{1} \frac{e^{icxt}}{t-a} \psi_j(t) \, dt. \tag{147}$$

Differentiating (147) with respect to x, we have

$$\frac{d}{dx} f(a,x) = ic \int_{-1}^{1} \frac{t e^{icxt}}{t-a} \psi_j(t) \, dt. \tag{148}$$

Multiplying (147) by ica, and subtracting it from (148), we obtain

$$\begin{aligned} \frac{d}{dx} f(a,x) - icaf(a,x) &= ic \int_{-1}^{1} e^{icxt} \, \psi_j(t) \, dt \\ &= ic\lambda_j \psi_j(x). \end{aligned} \tag{149}$$

In other words, f satisfies the differential equation

$$f'(x) - icaf(x) = ic\lambda_j \psi_j(x). \tag{150}$$

The standard "variation of parameter" calculation provides the solution to (150):

$$f(x) = ic\lambda_j \int_0^x e^{-ica(x-t)} \psi_j(t) \, dt + f(0) \, e^{icax}. \tag{151}$$

Introducing the notation

$$\mathcal{D} = \frac{1}{ic} \circ \frac{d}{dx} \tag{152}$$

(i.e. $\mathcal{D}$ is the product of multiplication by $1/ic$ and differentiation), we rewrite (146) as

$$\mathcal{D}^k(\psi_j)(x) = \frac{1}{\lambda_j}\int_{-1}^{1} t^k\, e^{icxt}\, \psi_j(t)dt; \tag{153}$$

for an arbitrary polynomial P (with real or complex coefficients),

$$P(\mathcal{D})(\psi_j)(x) = \frac{1}{\lambda_j}\int_{-1}^{1} P(t)\, e^{icxt}\, \psi_j(t)dt. \tag{154}$$

By the same token, the function ϕ defined by the formula

$$\phi(x) = \int_{-1}^{1} \frac{e^{icxt}}{P(t)}\psi_j(t)dt \tag{155}$$

satisfies the differential equation

$$P(\mathcal{D})(\phi)(x) = \lambda_m \psi_m(x). \tag{156}$$

The following lemma provides a recursion connecting the values of the k-th derivative of the function ψ_m with its derivatives of orders $k-1$, $k-2$, $k-3$, $k-4$.

Lemma 9.1 *For any positive real c, integer $m \geq 0$, and $x \in (-\infty, +\infty)$,*

$$\begin{aligned} &(1-x^2)\,\psi_m^{(k+2)}(x) - 2\,(k+1)\,x\,\psi_m^{(k+1)}(x) \\ &+ (\chi_m - k\,(k+1) - c^2\,x^2)\,\psi_m^{(k)}(x) \\ &- 2\,c^2\,k\,x\,\psi_m^{(k-1)}(x) - c^2\,k\,(k-1)\,\psi_m^{(k-2)}(x) = 0 \end{aligned} \tag{157}$$

for all $k \geq 2$. Furthermore,

$$\begin{aligned} &(1-x^2)\,\psi_m'''(x) - 4\,x\,\psi_m''(x) + (\chi_m - 2 - c^2\,x^2)\,\psi_m'(x) \\ &- 2\,c^2\,x\,\psi_m(x) = 0\,. \end{aligned} \tag{158}$$

In particular,

$$\begin{aligned} &-2\,(k+1)\,\psi_m^{(k+1)}(1) + (\chi_m - k\,(k+1) - c^2)\,\psi_m^{(k)}(1) \\ &- 2\,c^2\,k\,\psi_m^{(k-1)}(1) - c^2\,k\,(k-1)\,\psi_m^{(k-2)}(1) = 0 \end{aligned} \tag{159}$$

for all $k \geq 2$, and

$$-2\,\psi_m'(1) + (\chi_m - c^2)\,\psi_m(1) = 0\,, \tag{160}$$

$$-4\,\psi_m''(1) + (\chi_m - 2 - c^2)\,\psi_m'(1) - 2\,c^2\,\psi_m(1) = 0\,. \tag{161}$$

Furthermore, for all integer $m \geq 0$ and $k \geq 2$,

$$\psi_m^{(k+2)}(0) + (\chi_m - k\,(k+1))\,\psi_m^{(k)}(0)$$
$$- c^2\,k\,(k-1)\,\psi_m^{(k-2)}(0) = 0\,. \tag{162}$$

For all odd m,

$$\psi_m'''(0) + (\chi_m - 2)\,\psi_m'(0) = 0\,, \tag{163}$$

and for all even m,

$$\psi_m''(0) + \chi_m\,\psi_m(0) = 0\,. \tag{164}$$

Finally, for all integer $m \geq 0$, $k \geq 0$,

$$\psi_m(1) \neq 0\,, \tag{165}$$

$$\psi_{2m+1}^{(2k)}(0) = 0\,, \tag{166}$$

$$\psi_{2m}^{(2k+1)}(0) = 0\,. \tag{167}$$

Proof. All of the identities (157) – (164), (166), (167), are immediately obtained by repeated differentiation of (24).
In order to prove (165), we assume that

$$\psi_m(1) = 0 \tag{168}$$

for some integer $m \geq 0$, and observe that the combination of (168) with (159), (160), (161) implies that

$$\psi_m^{(k)}(1) = 0 \tag{169}$$

for all $k = 0, 1, 2, \ldots$. Due to the analyticity of $\psi_m(x)$ in the complex plane, this would imply that

$$\psi_m(x) = 0 \tag{170}$$

for all $x \in \mathbb{R}^1$.

$\square$

The following is an immediate consequence of the identity (160) of Lemma 9.1.

Corollary 9.2 *For all integer $m, n \geq 0$,*

$$\psi_m'(1) \cdot \psi_n(1) - \psi_n'(1) \cdot \psi_m(1) = (\chi_n - \chi_m) \cdot \psi_n(1) \cdot \psi_m(1)\,, \tag{171}$$

where $\chi_m, \chi_n \in \mathbb{R}$ are as defined in Theorem 2.6.

Theorem 3.1, in Section 3.1, gives formulae for the entries of matrices for differentiation of prolate series and for multiplication of prolate series by x. Matrices for any combination of differentiation and of multiplication by a polynomial can obviously be constructed from these two matrices; for instance, calling the differentiation matrix D, and the multiplication-by-x matrix X, the matrix for taking the second derivative of a prolate series, then multiplying it by $5 - x^2$, is equal to $(5I - X^2)D^2$.

In many cases, however, there are simpler formulae for the entries of such matrices, that is, for inner products of $\psi_j(x)$ with its derivatives and with polynomials. The following theorems establish several such formulae, as well as a few formulae for inner products which do not involve $\psi_j(x)$ itself but only its derivatives. We start with Theorem 3.1, restated here for consistency.

Theorem 9.3 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m = n \pmod 2$, then*

$$\int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\; dx = \int_{-1}^{1} x\, \psi_n(x)\, \psi_m(x)\; dx = 0. \tag{172}$$

If $m \neq n \pmod 2$, then

$$\int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\; dx = \frac{2\,\lambda_m^2}{\lambda_m^2 + \lambda_n^2}\, \psi_m(1)\, \psi_n(1)\,, \tag{173}$$

$$\int_{-1}^{1} x\, \psi_n(x)\, \psi_m(x)\; dx = \frac{2}{ic}\, \frac{\lambda_m \lambda_n}{\lambda_m^2 + \lambda_n^2}\, \psi_m(1)\, \psi_n(1)\,. \tag{174}$$

Theorem 9.4 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m \neq n \pmod 2$, then*

$$\int_{-1}^{1} x\, \psi_n'(x)\, \psi_m(x)\; dx = 0\,. \tag{175}$$

If $m = n \pmod 2$, then

$$\int_{-1}^{1} x\, \psi_n'(x)\, \psi_m(x)\; dx = \frac{\lambda_m}{\lambda_m + \lambda_n}\, \left(2\, \psi_m(1)\, \psi_n(1) - \delta_{mn}\right)\,. \tag{176}$$

Proof. Identity (175) is obvious since the functions ψ_j are alternately even and odd (see Theorem 2.4). In order to prove (176), we consider the integral

$$\begin{aligned} &\int_{-1}^{1} x\, \psi_n'(x)\, \psi_m(x)\; dx \\ &= \frac{1}{\lambda_n} \int_{-1}^{1} x \left(\int_{-1}^{1} e^{icxt}\, \psi_n(t)\; dt \right)_x' \psi_m(x)\; dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{ic}{\lambda_n} \int_{-1}^{1} x\,\psi_m(x) \left(\int_{-1}^{1} t\,\psi_n(t)\, e^{icxt}\, dt \right) dx \\
&= \frac{ic}{\lambda_n} \int_{-1}^{1} t \left(\int_{-1}^{1} x\,\psi_m(x)\, e^{icxt}\, dx \right) \psi_n(t)\, dt \\
&= \frac{\lambda_m}{\lambda_n} \int_{-1}^{1} t\,\psi_m'(t)\,\psi_n(t)\, dt\,.
\end{aligned}
$$

In other words,

$$
\int_{-1}^{1} x\,\psi_n'(x)\,\psi_m(x)\, dx = \frac{\lambda_m}{\lambda_n} \int_{-1}^{1} x\,\psi_m'(x)\,\psi_n(x)\, dx\,. \tag{177}
$$

On the other hand, integrating the left side of (177) by parts, we obtain

$$
\begin{aligned}
&\int_{-1}^{1} x\,\psi_n'(x)\,\psi_m(x)\, dx \\
&= 2\,\psi_m(1)\,\psi_n(1) - \int_{-1}^{1} \left(\psi_n(x)\,\psi_m'(x)\,x + \psi_n(x)\,\psi_m(x)\right) dx \\
&= 2\,\psi_m(1)\,\psi_n(1) - \int_{-1}^{1} x\,\psi_n(x)\,\psi_m'(x)\, dx - \delta_{mn}\,.
\end{aligned}
$$

Combining (177) and (178), we have

$$
\begin{aligned}
&\frac{\lambda_m}{\lambda_n} \int_{-1}^{1} x\,\psi_m'(x)\,\psi_n(x)\, dx \\
&= 2\psi_m(1)\,\psi_n(1) - \int_{-1}^{1} x\,\psi_m'(x)\,\psi_n(x)\, dx - \delta_{mn}\,,
\end{aligned}
$$

from which (176) follows directly. □

Theorem 9.5 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m \neq n \pmod 2$, then*

$$
\int_{-1}^{1} x^2\,\psi_n''(x)\,\psi_m(x)\, dx = 0\,. \tag{178}
$$

If $m = n \pmod 2$ and $m \neq n$, then

$$
\begin{aligned}
&\int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\, dx \\
&= \frac{2\lambda_n}{\lambda_m - \lambda_n} \left(\psi_n'(1)\,\psi_m(1) - \psi_m'(1)\,\psi_n(1)\right) \\
&\quad - \frac{4\lambda_n}{\lambda_n + \lambda_m}\,\psi_n(1)\,\psi_m(1)
\end{aligned} \tag{179}
$$

$$
\begin{aligned}
&= \frac{\lambda_n}{\lambda_m - \lambda_n} (\chi_n - \chi_m)\, \psi_n(1)\, \psi_m(1) \\
&\quad - \frac{4\lambda_n}{\lambda_n + \lambda_m}\, \psi_n(1)\, \psi_m(1),
\end{aligned} \tag{180}
$$

where $\chi_m, \chi_n \in \mathbb{R}$ are as defined in Theorem 2.6.

Proof. Clearly (178) is true, since the functions ψ_j are alternately even and odd. In order to prove (179) and (180), supposing that $m = n \pmod 2$ and $m \neq n$, we consider the integral

$$
\begin{aligned}
&\int_{-1}^{1} x^2\, \psi_n''(x)\, \psi_m(x)\; dx \\
&= \frac{1}{\lambda_n} \int_{-1}^{1} x^2 \cdot \left(\int_{-1}^{1} e^{icxt}\, \psi_n(t)\; dt \right)_x'' \psi_m(x)\; dx \\
&= -\frac{c^2}{\lambda_n} \int_{-1}^{1} \psi_m(x)\, x^2 \cdot \left(\int_{-1}^{1} t^2\, \psi_n(t)\, e^{icxt}\; dt \right)\; dx \\
&= -\frac{c^2}{\lambda_n} \int_{-1}^{1} \left(\int_{-1}^{1} \psi_m(x)\, x^2\, e^{icxt}\, dx \right) \psi_n(t)\, t^2\; dt \\
&= \frac{\lambda_m}{\lambda_n} \int_{-1}^{1} t^2\, \psi_n(t)\, \psi_m''(t)\; dt\,,
\end{aligned}
$$

which is summarized as

$$
\int_{-1}^{1} x^2\, \psi_n''(x)\, \psi_m(x)\; dx = \frac{\lambda_m}{\lambda_n} \int_{-1}^{1} x^2\, \psi_m''(x)\, \psi_n(x)\; dx\,. \tag{181}
$$

On the other hand, integrating the left side of (181) by parts, we have

$$
\begin{aligned}
&\int_{-1}^{1} x^2\, \psi_n''(x)\, \psi_m(x)\; dx \\
&= 2\, \psi_n'(1)\, \psi_m(1) - \int_{-1}^{1} \psi_n'(x)\, \left(\psi_m'(x)\, x^2 + 2\, x\, \psi_m(x) \right)\; dx \\
&= 2\, \psi_n'(1)\, \psi_m(1) - 2 \int_{-1}^{1} \psi_n'(x)\, \psi_m(x)\, x\; dx \\
&\quad - \int_{-1}^{1} \psi_n'(x)\, \psi_m'(x)\, x^2\; dx\,.
\end{aligned} \tag{182}
$$

Due to Theorem 9.4 and the fact that $m \neq n$, we immediately rewrite (182) as

$$
\begin{aligned}
&\int_{-1}^{1} x^2\, \psi_n''(x)\, \psi_m(x)\; dx \\
&= 2\psi_n'(1)\, \psi_m(1) - \frac{2\lambda_m}{\lambda_m + \lambda_n}\, 2\, \psi_n(1)\, \psi_m(1) \\
&\quad - \int_{-1}^{1} x^2\, \psi_n'(x)\, \psi_m'(x)\; dx\,,
\end{aligned} \tag{183}
$$

which we rewrite as

$$
\begin{aligned}
&\int_{-1}^{1} x^2\,\psi_n'(x)\,\psi_m'(x)\;dx \\
&= \;2\,\psi_n'(1)\,\psi_m(1) - \frac{4\,\lambda_m}{\lambda_m+\lambda_n}\,\psi_n(1)\,\psi_m(1) \\
&\quad - \int_{-1}^{1} x^2\,\psi_n''(x)\,\psi_m(x)\;dx\,. \qquad (184)
\end{aligned}
$$

Swapping m with n, we convert (184) into

$$
\begin{aligned}
&\int_{-1}^{1} x^2\,\psi_n'(x)\,\psi_m'(x)\;dx \\
&= \;2\,\psi_m'(1)\,\psi_n(1) - \frac{4\,\lambda_n}{\lambda_m+\lambda_n}\,\psi_n(1)\,\psi_m(1) \\
&\quad - \int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\;dx. \qquad (185)
\end{aligned}
$$

Combining (184) and (185), we obtain

$$
\begin{aligned}
&\int_{-1}^{1} x^2\,\psi_n''(x)\,\psi_m(x)\;dx - 2\,\psi_n'(1)\,\psi_m(1) + \frac{4\,\lambda_m}{\lambda_m+\lambda_n}\,\psi_n(1)\,\psi_m(1) \\
= &\int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\;dx - 2\,\psi_m'(1)\,\psi_n(1) + \frac{4\,\lambda_n}{\lambda_m+\lambda_n}\,\psi_n(1)\,\psi_m(1)\,, \qquad (186)
\end{aligned}
$$

which is obviously equivalent to

$$
\begin{aligned}
&\int_{-1}^{1} x^2\,\psi_n''(x)\,\psi_m(x)\;dx \\
&= \int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\;dx + 2\;(\psi_n'(1)\,\psi_m(1) - \psi_m'(1)\,\psi_n(1)) \\
&\quad + 4\,\frac{\lambda_n-\lambda_m}{\lambda_n+\lambda_m}\,\psi_n(1)\,\psi_m(1)\,.
\end{aligned}
$$

Finally, combining (181) with (187), we have

$$
\begin{aligned}
&\frac{\lambda_m}{\lambda_n}\int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\;dx \\
&= \int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\;dx + 2\;(\psi_n'(1)\,\psi_m(1) - \psi_m'(1)\,\psi_n(1)) \\
&\quad + 4\,\frac{\lambda_n-\lambda_m}{\lambda_n+\lambda_m}\,\psi_n(1)\,\psi_m(1)\,, \qquad (187)
\end{aligned}
$$

which is easily rewritten as

$$
\begin{aligned}
&\left(\frac{\lambda_m}{\lambda_n}-1\right)\int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\;dx \\
&= 2\,(\psi_n'(1)\,\psi_m(1)-\psi_m'(1)\,\psi_n(1)) \\
&\quad +4\,\frac{\lambda_n-\lambda_m}{\lambda_n+\lambda_m}\,\psi_n(1)\,\psi_m(1)\,,
\end{aligned}
$$

or

$$
\begin{aligned}
&\int_{-1}^{1} x^2\,\psi_m''(x)\,\psi_n(x)\;dx \\
&= \frac{2\,\lambda_n}{\lambda_m-\lambda_n}\,(\psi_n'(1)\,\psi_m(1)-\psi_m'(1)\,\psi_n(1)) \\
&\quad -\frac{4\,\lambda_n}{\lambda_n+\lambda_m}\,\psi_n(1)\,\psi_m(1)\,.
\end{aligned} \tag{188}
$$

We finally rewrite (188) as (180) using Corollary 9.2. □

The following theorem is an immediate consequence of combining the preceding theorem with equation (184) from its proof.

Theorem 9.6 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m \neq n \pmod 2$, then*

$$\int_{-1}^{1} x^2\,\psi_n'(x)\,\psi_m'(x)\;dx = 0\,. \tag{189}$$

If $m = n \pmod 2$ and $m \neq n$,

$$
\begin{aligned}
&\int_{-1}^{1} x^2\,\psi_m'(x)\,\psi_n'(x)\;dx \\
&= 2\,\psi_m'(1)\,\psi_n(1)+\frac{2\lambda_n}{\lambda_m-\lambda_n}\,(\psi_m'(1)\,\psi_n(1)-\psi_n'(1)\,\psi_m(1)) && (190)\\
&= 2\,\psi_n'(1)\,\psi_m(1)+\frac{2\lambda_m}{\lambda_n-\lambda_m}\,(\psi_n'(1)\,\psi_m(1)-\psi_m'(1)\,\psi_n(1)) && (191)\\
&= \psi_m(1)\,\psi_n(1)\left(\frac{\lambda_m\chi_m-\lambda_n\chi_n}{\lambda_m-\lambda_n}-c^2\right). && (192)
\end{aligned}
$$

Theorem 9.7 *Suppose that c is real and positive, and that the integers m and n are non-negative. If $m \neq n \pmod 2$, then*

$$\int_{-1}^{1}\psi_n(x)\,\psi_m''(x)\;dx = \int_{-1}^{1} x^2\,\psi_n(x)\,\psi_m(x)\;dx = 0 \tag{193}$$

If $m = n \pmod 2$ and $m \neq n$, then

$$\begin{aligned}
&\int_{-1}^{1} \psi_n(x)\, \psi_m''(x)\; dx \\
&= \frac{2\,\lambda_n^2}{\lambda_m^2 - \lambda_n^2}\, (\psi_n'(1)\, \psi_m(1) - \psi_n(1)\, \psi_m'(1)) && (194)\\
&= \frac{\lambda_n^2}{\lambda_m^2 - \lambda_n^2}\, (\chi_n - \chi_m)\, \psi_m(1)\, \psi_n(1), && (195)
\end{aligned}$$

$$\begin{aligned}
&\int_{-1}^{1} x^2\, \psi_n(x)\, \psi_m(x)\; dx \\
&= -\frac{2}{c^2}\, \frac{\lambda_m \lambda_n}{\lambda_m^2 - \lambda_n^2}\, (\psi_n'(1)\, \psi_m(1) - \psi_n(1)\, \psi_m'(1)) && (196)\\
&= -\frac{1}{c^2}\, \frac{\lambda_m \lambda_n}{\lambda_m^2 - \lambda_n^2}\, (\chi_n - \chi_m)\, \psi_m(1)\, \psi_n(1), && (197)
\end{aligned}$$

where $\chi_m, \chi_n \in \mathbb{R}$ are as defined in Theorem 2.6.

Proof. Identity (193) is obvious, since the functions ψ_j are alternately even and odd. In order to prove (194)–(197), we start with the expression

$$\lambda_n \psi_n''(x) = -c^2 \int_{-1}^{1} t^2\, e^{icxt}\, \psi_n(t)\; dt\,. \tag{198}$$

Taking the inner product of (198) with $\psi_m(x)$, we have

$$\begin{aligned}
&\lambda_n \int_{-1}^{1} \psi_n''(x)\, \psi_m(x)\; dx \\
&= -c^2 \int_{-1}^{1} \left(\int_{-1}^{1} t^2\, \psi_n(t)\, e^{icxt}\; dt \right) \psi_m(x)\; dx \\
&= -c^2 \int_{-1}^{1} t^2\, \psi_n(t) \left(\int_{-1}^{1} \psi_m(x)\, e^{icxt}\; dx \right) dt \\
&= -c^2 \lambda_m \int_{-1}^{1} t^2\, \psi_n(t)\, \psi_m(t)\; dt\,,
\end{aligned}$$

which we summarize as

$$\int_{-1}^{1} x^2\, \psi_n(x)\, \psi_m(x)\; dx = -\frac{1}{c^2}\, \frac{\lambda_n}{\lambda_m} \int_{-1}^{1} \psi_n''(x)\, \psi_m(x)\; dx\,. \tag{199}$$

Swapping n, m, we rewrite (199) in the form of

$$\begin{aligned}
&\int_{-1}^{1} x^2\, \psi_n(x)\, \psi_m(x) dx \\
&= -\frac{1}{c^2}\, \frac{\lambda_m}{\lambda_n} \int_{-1}^{1} \psi_m''(x)\, \psi_n(x)\; dx\,. && (200)
\end{aligned}$$

Combining (199) and (200), we get

$$\int_{-1}^{1} \psi_n''(x)\,\psi_m(x)\;dx = \frac{\lambda_m^2}{\lambda_n^2}\int_{-1}^{1}\;\psi_m''(x)\,\psi_n(x)\;dx\,. \tag{201}$$

On the other hand, integrating the left side of (201) by parts, we have

$$\begin{aligned}
&\int_{-1}^{1} \psi_n''(x)\,\psi_m(x)\;dx \\
&= \psi_n'(1)\,\psi_m(1) - \psi_n'(-1)\,\psi_m(-1) - \int_{-1}^{1}\;\psi_n'(x)\,\psi_m'(x)\;dx \\
&= 2\,\psi_n'(1)\,\psi_m(1) - (\psi_n(1)\,\psi_m'(1) - \psi_n(-1)\,\psi_m'(-1)) \\
&\quad + \int_{-1}^{1}\;\psi_n(x)\,\psi_m''(x)\;dx.
\end{aligned} \tag{202}$$

We rewrite (202) in the form of

$$\begin{aligned}
&\int_{-1}^{1} \psi_n''(x)\,\psi_m(x)\;dx \\
&= 2\,(\psi_n'(1)\,\psi_m(1) - \psi_n(1)\,\psi_m'(1)) + \int_{-1}^{1}\psi_n(x)\,\psi_m''(x)\;dx\,.
\end{aligned}$$

We combine (201) and (203) and get

$$\begin{aligned}
&\left(\frac{\lambda_m^2}{\lambda_n^2} - 1\right)\int_{-1}^{1}\;\psi_n(x)\,\psi_m''(x)\;dx \\
&= 2\;(\psi_n'(1)\,\psi_m(1) - \psi_n(1)\,\psi_m'(1))\;.
\end{aligned} \tag{203}$$

Since $m \neq n$, we easily rewrite (203) as (194). We obtain expression (196) by combining (200) and (194). The identities (195), (197) follow from (194), (196) immediately due to Corollary 9.2. □

Theorem 9.8 *Suppose that c is real and positive, and that the integers m and n are non-negative. Let*

$$\Psi_n(y) = \int_0^y \psi_n(x)\;dx\,. \tag{204}$$

If n is odd and m is even, then

$$\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\,\psi_m(t)\;dt \tag{205}$$

$$= i\,c\,\frac{2\,\lambda_m\,\lambda_n}{\lambda_n^2+\lambda_m^2}\,\Psi_n(1)\,\Psi_m(1) \tag{206}$$

$$+\,2\,\frac{\lambda_m}{\lambda_n^2+\lambda_m^2}\,\Psi_m(1)\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt. \tag{207}$$

If $m = n \pmod 2$, *then*

$$\int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\, \psi_m(t)\; dt = 0\,. \tag{208}$$

Proof. We start with the identity

$$\lambda_n\, \psi_n(x) = \int_{-1}^{1} e^{icxt}\, \psi_n(t)\; dt\,. \tag{209}$$

Integrating (209) with respect to x, we have

$$\begin{aligned}
&\lambda_n \int_0^y \psi_n(x)\, dx \\
&= \int_0^y \left(\int_{-1}^{1} e^{icxt}\, \psi_n(t)\; dt \right)\; dx && (210)\\
&= \int_{-1}^{1} \psi_n(t) \int_0^y e^{ixct}\; dx\, dt && (211)\\
&= \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\, e^{icyt}\; dt - \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt\,, && (212)
\end{aligned}$$

which we summarize as

$$\lambda_n\, \Psi_n(y) = \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\, e^{icyt}\; dt - \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt\,. \tag{213}$$

Taking the inner product of (213) and $\psi_m(y)$, we obtain

$$\begin{aligned}
&\lambda_n \int_{-1}^{1} \Psi_n(y)\, \psi_m(y)\; dy \\
&= \frac{1}{i\,c} \int_{-1}^{1} \psi_m(y) \cdot \left(\int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\, e^{icyt}\; dt \right)\; dy \\
&\quad - \frac{1}{i\,c} \int_{-1}^{1} \psi_m(y) \cdot \left(\int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \right)\; dy && (214)\\
&= \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t) \cdot \left(\int_{-1}^{1} e^{icyt}\, \psi_m(y)\; dy \right)\; dt \\
&\quad - \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(y)\; dy && (215)\\
&= \frac{\lambda_m}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\, \psi_m(t)\; dt \\
&\quad - \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(y)\; dy\,, && (216)
\end{aligned}$$

which we summarize as

$$
\begin{aligned}
&\int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\, \psi_m(t)\; dt \\
&= \; i\, c\, \frac{\lambda_n}{\lambda_m} \int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt \\
&\quad + \frac{1}{\lambda_m} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(y)\; dy
\end{aligned} \tag{217}
$$

Exchanging m with n, we convert (217) into

$$
\begin{aligned}
&\int_{-1}^{1} \frac{1}{t}\, \psi_m(t)\, \psi_n(t)\; dt \\
&= \; i\, c\, \frac{\lambda_m}{\lambda_n} \int_{-1}^{1} \Psi_m(t)\, \psi_n(t)\; dt \\
&\quad + \frac{1}{\lambda_n} \int_{-1}^{1} \frac{1}{t}\, \psi_m(t)\; dt \cdot \int_{-1}^{1} \psi_n(y)\; dy\,,
\end{aligned} \tag{218}
$$

and combining (217), (218), we get

$$
\begin{aligned}
&\frac{\lambda_n}{\lambda_m}\, i\, c \int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt - \frac{\lambda_m}{\lambda_n}\, i\, c \int_{-1}^{1} \Psi_m(t)\, \psi_n(t)\; dt \\
&= \; \frac{1}{\lambda_n} \int_{-1}^{1} \frac{1}{t}\, \psi_m(t)\; dt \cdot \int_{-1}^{1} \psi_n(t)\; dt \\
&\quad - \frac{1}{\lambda_m} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(t)\; dt\,.
\end{aligned} \tag{219}
$$

Suppose that m is even and n is odd; then the first product in the right hand side of (219) is zero, so

$$
\begin{aligned}
&\frac{\lambda_n}{\lambda_m}\, i\, c \int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt - \frac{\lambda_m}{\lambda_n}\, i\, c \int_{-1}^{1} \Psi_m(t)\, \psi_n(t)\; dt \\
&= \; - \frac{1}{\lambda_m} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(t)\; dt\,,
\end{aligned} \tag{220}
$$

which is equivalent to

$$
\begin{aligned}
&\int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt \\
&= \; \frac{\lambda_m^2}{\lambda_n^2} \int_{-1}^{1} \Psi_m(t)\, \psi_n(t)\; dt \\
&\quad - \frac{1}{\lambda_n}\, \frac{1}{i\, c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(t)\; dt\,,
\end{aligned} \tag{221}
$$

or

$$\begin{aligned}
&\int_{-1}^{1} \Psi_m(t)\, \psi_n(t)\; dt \\
&= \frac{\lambda_n^2}{\lambda_m^2} \int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt \\
&\quad + \frac{\lambda_n}{\lambda_m^2} \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(t)\; dt\,. \qquad (222)
\end{aligned}$$

On the other hand, integrating the left side of (222) by parts, we obtain

$$\begin{aligned}
&\int_{-1}^{1} \Psi_m(t)\, \psi_n(t)\; dt \\
&= \Psi_n(1)\, \Psi_m(1) - \Psi_n(-1)\, \Psi_m(-1) - \int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt\,. \qquad (223)
\end{aligned}$$

Since the product $\Psi_m(x)\, \Psi_n(x)$ is an odd function when $m \neq n \pmod 2$, we rewrite (223) as

$$\begin{aligned}
&\int_{-1}^{1} \Psi_m(t)\, \psi_n(t)\; dt \\
&= 2\, \Psi_n(1)\, \Psi_m(1) - \int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt\,. \qquad (224)
\end{aligned}$$

The combination of (222) and (224) implies that

$$\begin{aligned}
&\int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt + \frac{\lambda_n^2}{\lambda_m^2} \int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt \\
&= 2\, \Psi_n(1)\, \Psi_m(1) - \frac{\lambda_n}{\lambda_m^2} \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(t)\; dt\,, \qquad (225)
\end{aligned}$$

or

$$\begin{aligned}
&\frac{\lambda_m^2 + \lambda_n^2}{\lambda_m^2} \int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt \\
&= 2\, \Psi_n(1)\, \Psi_m(1) - \frac{\lambda_n}{\lambda_m^2} \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(t)\; dt\,, \qquad (226)
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\int_{-1}^{1} \Psi_n(t)\, \psi_m(t)\; dt \\
&= \frac{2\,\lambda_m^2}{\lambda_n^2 + \lambda_m^2}\, \Psi_n(1)\, \Psi_m(1) \\
&\quad - \frac{\lambda_n}{\lambda_n^2 + \lambda_m^2} \frac{1}{i\,c} \int_{-1}^{1} \frac{1}{t}\, \psi_n(t)\; dt \cdot \int_{-1}^{1} \psi_m(t)\; dt\,. \qquad (227)
\end{aligned}$$

Finally, combining (217) and (227), we have

$$\int_{-1}^{1} \frac{1}{t}\,\psi_n(t)\,\psi_m(t)\;dt$$
$$= \; i\,c\,\frac{2\,\lambda_m\,\lambda_n}{\lambda_n^2+\lambda_m^2}\,\Psi_n(1)\,\Psi_m(1) + \frac{\lambda_m}{\lambda_n^2+\lambda_m^2}\int_{-1}^{1}\frac{1}{t}\,\psi_n(t)\;dt\cdot\int_{-1}^{1}\psi_m(t)\;dt\,. \tag{228}$$

Equation (208) is easily proven since the product $\frac{1}{t}\,\psi_m(x)\,\psi_n(x)$ is an odd function whenever $m = n \pmod 2$. □

The above theorems do not use much of the detailed structure of the integral operators of which the functions $\{\psi_j\}$ are eigenfunctions. Thus many of them generalize easily to the case of an operator $L : L^2[0,1] \to L^2[0,1]$ defined via the formula

$$L(\psi)(x) = \int_0^1 K(xt)\,\psi(t)\;dt, \tag{229}$$

for some function $K : [0,1] \to \mathbb{C}$; the following theorem is an example of this.

Theorem 9.9 *Let λ_1, λ_2 be two eigenvalues of the operator L defined by (229), that is,*

$$\int_0^1 K(xt)\,\psi_1(t)\;dt \;=\; \lambda_1\psi_1(x), \tag{230}$$
$$\int_0^1 K(xt)\,\psi_2(t)\;dt \;=\; \lambda_2\psi_2(x). \tag{231}$$

Then

$$\frac{\lambda_2}{\lambda_1} = \frac{\displaystyle\int_0^1 x\,\psi_1'(x)\,\psi_2(x)\;dx}{\displaystyle\int_0^1 x\,\psi_2'(x)\,\psi_1(x)\;dx}, \tag{232}$$

provided that neither λ_1 nor the denominator of the right hand side of (232) is zero.

Proof. Differentiating (230), (231) with respect to x, we get

$$\int_0^1 t\,K'(xt)\,\psi_1(t)\;dt \;=\; \lambda_1\psi_1'(x), \tag{233}$$
$$\int_0^1 t\,K'(xt)\,\psi_2(t)\;dt \;=\; \lambda_2\psi_2'(x). \tag{234}$$

Multiplying (233) by $x\,\psi_2(x)$, we have

$$\lambda_1\,x\,\psi_1'(x)\,\psi_2(x) = x\,\psi_2(x)\int_0^1 t\,K'(xt)\,\psi_1(t)\;dt. \tag{235}$$

Integrating on the interval $[0,1]$, we obtain

$$\lambda_1 \int_0^1 x\,\psi_1'(x)\,\psi_2(x)\;dx \;=\; \int_0^1 x\,\psi_2(x) \int_0^1 t\,K'(xt)\,\psi_1(t)\;dt\,dx \tag{236}$$

$$= \int_0^1 t\,\psi_1(t) \int_0^1 x\,K'(xt)\,\psi_2(x)\;dx\,dt. \tag{237}$$

Renaming the variables of integration on the right hand side from x to t and vice versa, we get

$$\lambda_1 \int_0^1 x\,\psi_1'(x)\,\psi_2(x)\;dx \;=\; \int_0^1 x\,\psi_1(x) \int_0^1 t\,K'(xt)\,\psi_2(t)\;dt\,dx. \tag{238}$$

Substituting (234) into (238), we obtain

$$\lambda_1 \int_0^1 x\,\psi_1'(x)\,\psi_2(x)\;dx \;=\; \lambda_2 \int_0^1 x\,\psi_1(x)\,\psi_2'(x)\,dx, \tag{239}$$

from which (232) follows immediately, as does its caveat. □

The following theorem establishes the relation between the norm of each function ψ_j on $[-1,1]$ (which in this paper is taken to be one), and its norm on $(-\infty,\infty)$.

Theorem 9.10 *Suppose that c is real and positive, and that the integer n is non-negative. Then*

$$\int_{-\infty}^{\infty} \psi_n^2(x)\;dx = \frac{1}{\mu_n}. \tag{240}$$

where μ_n is given by (21).

Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi_n^2(x)\;dx &= \int_{-\infty}^{\infty} \left(\frac{1}{\pi\mu_n}\int_{-1}^{1}\psi_n(t)\,\frac{\sin(c\cdot(x-t))}{x-t}\;dt\right)\psi_n(x)\;dx \\
&= \frac{1}{\mu_n}\int_{-1}^{1}\psi_n(t)\cdot\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(c\cdot(x-t))}{x-t}\,\psi_n(x)\;dx\right)\;dt \\
&= \frac{1}{\mu_n}\int_{-1}^{1}\psi_n^2(t)\;dt \\
&= \frac{1}{\mu_n}.
\end{aligned}$$

□

The following theorem extends Theorem (9.10) to any band-limited function with band limit c.

Theorem 9.11 *Suppose that c is real and positive, that the integer n is non-negative, and that $f : \mathbb{R} \to \mathbb{C}$ is a band-limited function with band limit c. Then*

$$\int_{-\infty}^{\infty} \psi_n(x)\, f(x)\; dx = \frac{1}{\mu_n} \int_{-1}^{1} \psi_n(x)\, f(x)\; dx. \tag{241}$$

Proof.

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi_n(x)\, f(x)\; dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\pi \mu_n} \int_{-1}^{1} \frac{\sin(c\cdot(x-t))}{x-t}\, \psi_n(t)\; dt \right) f(x)\; dx \\
&= \frac{1}{\mu_n} \int_{-1}^{1} \psi_n(t) \cdot \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(c\cdot(x-t))}{x-t}\, f(x)\; dx \right) dt \\
&= \frac{1}{\mu_n} \int_{-1}^{1} \psi_n(t)\, f(t)\; dt\,.
\end{aligned}$$

□

Theorem 9.12 *Suppose that c is real and positive, and that the integer n is non-negative. Then*

$$\int_{-\infty}^{\infty} e^{icxt} \psi_m(t) dt = \begin{cases} \dfrac{\lambda_m}{\mu_m} \psi_m(x), & \text{if } -1 < x < 1, \\ \\ 0, & \text{if } x > 1 \text{ or } x < -1. \end{cases} \tag{242}$$

Proof. Since ψ_m is an eigenfunction of the operator Q_c defined in (19), and μ_m is the corresponding eigenvalue,

$$\mu_m \psi_m(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{sin(c\cdot(x-u))}{x-u}\, \psi_m(u)\; du. \tag{243}$$

Thus

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{icxt}\, \psi_m(t)\; dt \\
&= \frac{1}{\mu_m} \int_{-\infty}^{\infty} e^{icxt} \left(\frac{1}{\pi} \int_{-1}^{1} \frac{sin(c\cdot(x-u))}{x-u}\, \psi_m(u)\; du \right) dt &\quad (244)\\
&= \frac{1}{\mu_m} \int_{-1}^{1} \psi_m(u) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{sin(c\cdot(x-u))}{x-u}\, e^{icxt}\; dt \right) du &\quad (245)
\end{aligned}$$

Since the innermost integral is the orthogonal projection operator onto the space of functions of band limit c on $(-\infty, \infty)$, applied to the function e^{icxt}, it follows that:

$$\int_{-\infty}^{\infty} e^{icxt}\, \psi_m(t)\; dt$$

$$= \frac{1}{\mu_m} \int_{-1}^{1} \psi_m(u) \left(\begin{Bmatrix} e^{icxu}, & \text{if } -1 < x < 1, \\ 0, & \text{if } x > 1 \text{ or } x < -1 \end{Bmatrix} \right) du \tag{246}$$

$$= \begin{cases} \dfrac{1}{\mu_m} \displaystyle\int_{-1}^{1} \psi_m(u)\, e^{icxu}\, du, & \text{if } -1 < x < 1, \\ \\ 0, & \text{if } x > 1 \text{ or } x < -1, \end{cases} \tag{247}$$

from which (242) follows immediately. □

The following five theorems establish formulae for the derivatives of prolate functions and their associated eigenvalues with respect to c.

Theorem 9.13 *For all positive real c and non-negative integer m,*

$$\frac{\partial \lambda_m}{\partial c} = \lambda_m \frac{2\, \psi_m^2(1) - 1}{2\, c}. \tag{248}$$

Proof. We start with

$$\lambda_m \psi_m(x) = \int_{-1}^{1} e^{icxt}\, \psi_m(t)\; dt\,. \tag{249}$$

Differentiating (249) with respect to c, we obtain

$$\frac{\partial \lambda_m}{\partial c}\, \psi_m(x) + \lambda_m \frac{\partial \psi_m(x)}{\partial c}$$

$$= \int_{-1}^{1} i\, x\, t\, e^{icxt}\, \psi_m(t)\; dt + \int_{-1}^{1} e^{icxt}\, \frac{\partial \psi_m(t)}{\partial c}\; dt\,. \tag{250}$$

Multiplying by $\psi_m(x)$ on both sides of (250), and integrating on the interval $[-1, 1]$, we get

$$\int_{-1}^{1} \psi_m(x) \left(\frac{\partial \lambda_m}{\partial c}\, \psi_m(x) + \lambda_m \frac{\partial \psi_m(x)}{\partial c} \right) dx \tag{251}$$

$$= \int_{-1}^{1} \psi_m(x) \int_{-1}^{1} i\, x\, t\, e^{icxt}\, \psi_m(t)\; dt\; dx$$

$$+ \int_{-1}^{1} \psi_m(x) \int_{-1}^{1} e^{icxt}\, \frac{\partial \psi_m(t)}{\partial c}\; dt\; dx\,,$$

which we rewrite as

$$\frac{\partial \lambda_m}{\partial c} + \lambda_m \int_{-1}^{1} \frac{\partial \psi_m(x)}{\partial c}\, \psi_m(x)\; dx$$
$$= \int_{-1}^{1} i\, t\, \psi_m(t) \int_{-1}^{1} e^{icxt}\, x\, \psi_m(x)\; dx\; dt$$
$$+ \int_{-1}^{1} \frac{\partial \psi_m(t)}{\partial c} \int_{-1}^{1} e^{icxt}\, \psi_m(x)\; dx\; dt \tag{252}$$
$$= \lambda_m \int_{-1}^{1} i\, t\, \psi_m(t)\, \frac{1}{i\, c}\, \frac{\partial \psi_m(t)}{\partial t}\; dt$$
$$+ \lambda_m \int_{-1}^{1} \frac{\partial \psi_m(t)}{\partial c}\, \psi_m(t)\; dt\,, \tag{253}$$

which we summarize as

$$\frac{\partial \lambda_m}{\partial c} = \frac{\lambda_m}{c} \int_{-1}^{1} t\, \psi_m(t)\, \frac{\partial \psi_m(t)}{\partial t}\; dt\,. \tag{254}$$

On the other hand, integrating the right-hand side of (254) by parts, we have

$$\int_{-1}^{1} t\, \psi_m(t)\, \frac{\partial \psi_m(t)}{\partial t}\; dt \tag{255}$$
$$= \psi_m^2(1) + \psi_m^2(-1) - 1 - \int_{-1}^{1} \psi_m(t)\, t\, \frac{\partial \psi_m(t)}{\partial t}\; dt\,,$$

which we rewrite as

$$\int_{-1}^{1} t\, \psi_m(t)\, \frac{\partial \psi_m(t)}{\partial t}\; dt = \psi_m^2(1) - \frac{1}{2}\,. \tag{256}$$

Finally, substituting (256) into (254), we get

$$\frac{\partial \lambda_m}{\partial c} = \lambda_m\, \frac{2\, \psi_m^2(1) - 1}{2\, c}\,. \tag{257}$$

□

Theorem 9.14 *For any positive real c and non-negative integer m,*

$$\frac{\partial \mu_m}{\partial c} = \frac{2}{c}\, \mu_m\, \psi_m^2(1)\,. \tag{258}$$

Proof. We start with the identity

$$\mu_m = \frac{2c}{\pi} \overline{\lambda}_m \, \lambda_m. \tag{259}$$

Differentiating (259) with respect to c, we get

$$\frac{\partial \mu_m}{\partial c} = \frac{2c}{\pi} \left(\overline{\lambda}_m \, \frac{\partial \lambda_m}{\partial c} + \lambda_m \, \frac{\partial \overline{\lambda}_m}{\partial c} \right) + \frac{2}{\pi} \overline{\lambda}_m \, \lambda_m \, . \tag{260}$$

Substituting Lemma 9.13 into (260), we get

$$\begin{aligned} \frac{\partial \mu_m}{\partial c} &= \frac{2c}{\pi} \cdot 2 \, \overline{\lambda}_m \, \lambda_m \, \frac{2 \, \psi_m^2(1) - 1}{2 \, c} + \frac{2}{\pi} \overline{\lambda}_m \, \lambda_m & (261) \\ &= 2 \, \mu_m \, \frac{2 \, \psi_m^2(1) - 1}{2 \, c} + \frac{1}{c} \, \mu_m \\ &= \frac{2}{c} \, \mu_m \, \psi_m^2(1) - \frac{1}{c} \, \mu_m + \frac{1}{c} \, \mu_m \\ &= \frac{2}{c} \, \mu_m \, \psi_m^2(1) \, . & (262) \end{aligned}$$

□

The following theorem immediately follows from Theorems 9.13 and 9.14.

Theorem 9.15 *For all positive real c and non-negative integer m, n,*

$$\left(\frac{\lambda_m}{\lambda_n} \right)' = \frac{\lambda_m}{\lambda_n} \frac{1}{c} \left(\psi_m^2(1) - \psi_n^2(1) \right) , \tag{263}$$

$$\left(\frac{\mu_m}{\mu_n} \right)' = \frac{\mu_m}{\mu_n} \frac{2}{c} \left(\psi_m^2(1) - \psi_n^2(1) \right) . \tag{264}$$

Theorem 9.16 *Suppose that c is real and positive, and the integers m, n are non-negative. If $m \neq n$, then*

$$\int_{-1}^{1} \psi_m(t) \, \frac{\partial \psi_n}{\partial c}(t) \; dt = -\frac{2}{c} \frac{\lambda_n \, \lambda_m}{\lambda_m^2 - \lambda_n^2} \, \psi_m(1) \, \psi_n(1) \, . \tag{265}$$

If $m = n$, then

$$\int_{-1}^{1} \psi_m(t) \, \frac{\partial \psi_n}{\partial c}(t) \; dt = 0. \tag{266}$$

Proof. Since the norm of ψ_n on $[-1,1]$ remains constant as c varies, ψ_n must be orthogonal on $[-1,1]$ to its own derivative with respect to c, which immediately yields (266). To establish (265), we start with the identity

$$\lambda_n \psi_n(x) = \int_{-1}^{1} e^{icxt}\, \psi_n(t)\; dt\,. \tag{267}$$

Differentiating (267) with respect to c, we get

$$\begin{aligned} &\frac{\partial \lambda_n}{\partial c}\, \psi_n(x) + \lambda_n\, \frac{\partial \psi_n}{\partial c} \\ &= \int_{-1}^{1} \left(i\, x\, t\, e^{icxt}\, \psi_n(t) + e^{icxt}\, \frac{\partial \psi_n(t)}{\partial c} \right)\; dt\,. \end{aligned} \tag{268}$$

Multiplying both sides of (268) by $\psi_m(x)$ and integrating with respect to x, we have

$$\begin{aligned} &\lambda_n \int_{-1}^{1} \psi_m(x)\, \frac{\partial \psi_n(x)}{\partial c}\; dx \\ &= \frac{\lambda_n}{c} \int_{-1}^{1} x\, \psi_n'(x)\, \psi_m(x)\; dx + \lambda_m \int_{-1}^{1} \psi_m(t)\, \frac{\partial \psi_n(t)}{\partial c}\; dt\,, \end{aligned} \tag{269}$$

which, using (176), we rewrite as

$$\begin{aligned} &(\lambda_n - \lambda_m) \int_{-1}^{1} \psi_m(t)\, \frac{\partial \psi_n(t)}{\partial c}\; dt \\ &= \frac{\lambda_n}{c}\, \frac{\lambda_m}{\lambda_m + \lambda_n}\, (2\, \psi_m(1)\, \psi_n(1) - \delta_{mn})\,. \end{aligned} \tag{270}$$

Assuming that $m \neq n$, and dividing by $\lambda_n - \lambda_m$, we then get (265). □

Theorem 9.17 *Suppose that c is real and positive, and the integer m is non-negative. Then*

$$\frac{\partial \chi_m}{\partial c} = 2c \int_{-1}^{1} x^2 \psi_m^2(x). \tag{271}$$

Proof. Due to Theorem 2.6,

$$(1 - x^2)\psi_m''(x) - 2x\psi_m'(x) + (\chi_m - c^2x^2)\, \psi_m(x) = 0. \tag{272}$$

Making the infinitesimal changes $c = c + h$, $\chi_m = \chi_m + \varepsilon$, and $\psi_m(x) = \psi_m(x) + \delta(x)$, this becomes

$$\begin{aligned} &(1 - x^2) \cdot (\psi_m''(x) + \delta''(x)) - 2x \cdot (\psi_m'(x) + \delta'(x)) \\ &\quad + (\chi_m + \varepsilon - (c+h)^2 x^2) \cdot (\psi_m(x) + \delta(x)) = 0. \end{aligned} \tag{273}$$

Expanding each term, discarding infinitesimals of the second order or greater (that is, products of two or more of the quantities h, ε, and $\delta(x)$), and subtracting (272), we get

$$(1-x^2)\,\delta''(x) - 2x\delta'(x) + (\chi_m - c^2x^2)\,\delta(x) + (\varepsilon - 2chx^2)\psi_m(x) = 0. \tag{274}$$

Let the self-adjoint differential operator L be defined by the formula

$$L(f)(x) = (1-x^2)f''(x) - 2xf'(x) + (\chi_m - c^2x^2)f(x). \tag{275}$$

Then, multiplying (274) by $\psi_m(x)/h$ and integrating on $[-1,1]$, we get

$$\int_{-1}^{1} L(\frac{\partial\psi_m}{\partial c})(x)\,\psi_m(x)\;dx + \frac{\varepsilon}{h} - \int_{-1}^{1} 2cx^2\psi_m^2(x) = 0. \tag{276}$$

Now $\frac{\varepsilon}{h} = \frac{\partial\chi_m}{\partial c}$. In addition, since L is self-adjoint,

$$\int_{-1}^{1} L(\frac{\partial\psi_m}{\partial c})(x)\,\psi_m(x)\;dx = \int_{-1}^{1} \frac{\partial\psi_m}{\partial c}(x)\,L(\psi_m)(x)\;dx. \tag{277}$$

But due to (272), $L(\psi_m)(x) = 0$ for all $x \in [-1,1]$, so the integral (277) is zero. Thus (276) becomes

$$\frac{\partial\chi_m}{\partial c} = 2c\int_{-1}^{1} x^2\psi_m^2(x). \tag{278}$$

□

10 Generalizations and Conclusions

In this paper, we design quadrature rules for band-limited functions, based on the properties of Prolate Spheroidal Wave Functions (PSWFs), and the connections of the latter with certain fundamental integral operators (see (17), (19) in Section 2.5). The quadratures are a surprisingly close analogue for band-limited functions of Gaussian quadratures for polynomials, in that they have positive weights, are optimal in the appropriately defined sense, and their nodes, when used for approximation (as opposed to integration), result in extremely efficient interpolation formulae. Thus, Sections 5-7 of this paper can be viewed as reproducing for band-limited functions much of the standard polynomial-based approximation theory (for which see, for example, [24]). Generally, there is a striking analogy between the band-limited functions and polynomials.

Obviously, there are certain differences between the resulting apparatus and the standard numerical analysis. To start with, where the classical techniques are optimal for polynomials, the approach of this paper is optimal for band-limited functions; whenever the functions to be dealt with are naturally represented by trigonometric expansions on finite intervals, our quadrature and interpolation formulae tend to be more efficient than those based on the polynomials. When the functions to be dealt with are naturally represented by polynomials, the classical approach is more efficient; however, many physical phenomena involve bend-limited functions, and very few involve polynomials.

Qualitatively, the quadrature (and interpolation) nodes obtained in this paper behave like a compromise between the Gaussian nodes and the equispaced ones: near the middle of the interval, they are very nearly equispaced, and near the ends, they concentrate somewhat, but much less than the Gaussian (or Chebychev) nodes do. For large c, the distance between nodes near the ends of the interval is of the order $\frac{1}{c^{3/2}}$, with the total number of nodes close to $\frac{c}{\pi}$. In contrast, the distance between the Gaussian nodes near the ends of the interval is of the order $\frac{1}{n^2}$, with n the total number of nodes. A closely related phenomenon is the reduced norm of the differentiation operator based on the prolate expansions: for an $n-$point differentiation formula, the norm is of the order $n^{3/2}$, as opposed to n^2 for polynomial-based spectral differentiation. Thus, PSWFs are likely to be a better tool for the design of spectral and pseudo-spectral techniques than the orthogonal polynomials and related functions.

Much of the analytical apparatus we use was developed more than 30 years ago (see [20]-[21], [17], [18]); the fundamental importance of these results in certain areas of electrical engineering and physics has also been understood for a long time. However, there appears to have been no prior attempt made to view band-limited functions as a source of *numerical* algorithms. Generally, there is a fairly limited amount of information in the literature about the PSWFs, especially when compared to the wealth of facts on many other special functions. Section 9 of this paper is an attempt to remedy this situation to a small degree.

The apparatus built in this paper is a strictly one-dimensional one. Obviously, one can construct discretizations of rectangles, cubes, etc. by using direct products of one-dimensional grids; the resulting numerical algorithms are satisfactory but not optimal. Furthermore, representation of band-limited functions on regions in higher dimensions is of both theoretical and engineering interest. Obvious applications include seismic data collection and processing, antenna theory, NMR imaging, and many others. When the region of interest is a sphere, most of the necessary analytical apparatus can be found in [21]. At the present time, we have constructed and implemented somewhat rudimentary versions of the relevant numerical algorithms; we are conducting numerical experiments with these, and will report the results at a later date. A much more difficult set of questions is presented by the structure of band-limited functions on more general regions.

References

[1] C.J. Bouwkamp, *On Spheroidal Wave Functions of Order Zero*, J. Math. Phys., 26, 1947, pp. 79-92.

[2] H. Cheng, N. Yarvin, V. Rokhlin, *Non-Linear Optimization, Quadrature, and Interpolation*, Yale University Technical Report, YALEU/DCS/RR-1169, 1998, to appear in the SIAM Journal of Non-linear Optimization.

[3] L. Debnath, *Integral Transforms and Their Applications*, CRC Press, New York, 1995.

[4] G. Gripenberg, S.O. Londen, O. Staffans, *Volterra Integral and Functional Equations, Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 1990.

[5] C. Flammer, *Spheroidal Wave Functions*, Stanford University Press, Stanford, Ca, 1956.

[6] F. Gantmacher and M. Krein, *Oscillation matrices and kernels and small oscillations of mechanical systems*, 2nd ed., Gosudarstv. Izdat, Tehn-Teor. Lit., Moscow, 1950 (Russian).

[7] F. A. Grünbaum, *Toeplitz Matrices Commuting With Tridiagonal Matrices*, J. Linear Alg. and Appl., 40, (1981).

[8] F. A. Grünbaum, *Eigenvectors of a Toeplitz Matrix: Discrete Version of the Prolate Spheroidal Wave Functions*, SIAM J. Alg. Disc. Meth., 2(1981).

[9] F. A. Grünbaum, L. Longhi, M. Perlstadt, *Differential Operators Commuting with Finite Convolution Integral Operators: Some Non-Abelian Examples*, SIAM J. Appl. Math. 42(1982).

[10] S. Karlin, *The Existence of Eigenvalues for Integral Operators*, Trans. Am. Math. Soc. v. 113, pp. 1-17 (1964).

[11] S. Karlin, and W. J. Studden, *Tchebycheff Systems with Applications In Analysis And Statistics*, John Wiley (Interscience), New York, 1966.

[12] M. G. Krein, *The Ideas of P. L. Chebyshev and A. A. Markov in the Theory Of Limiting Values Of Integrals*, American Mathematical Society Translations, Ser. 2, Vol. 12, 1959, pp. 1-122.

[13] J. MA, V. ROKHLIN, AND S. WANDZURA, *Generalized Gaussian Quadratures For Systems of Arbitrary Functions*, SIAM Journal of Numerical Analysis, v. 33, No. 3, pp. 971-996, 1996.

[14] A. A. MARKOV, *On the limiting values of integrals in connection with interpolation*, Zap. Imp. Akad. Nauk. Fiz.-Mat. Otd. (8) 6 (1898), no.5 (Russian), pp. 146-230 of [15].

[15] A. A. MARKOV, *Selected papers on continued fractions and the theory of functions deviating least from zero*, OGIZ, Moscow-Leningrad, 1948 (Russian).

[16] P.M. Morse, H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953.

[17] H.J. Landau, H.O. Pollak, *Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty - II*, The Bell System Technical Journal, January 1961.

[18] H.J. Landau, H.O. Pollak, *Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty - III: The Dimension of Space of Essentially Time- and Band-Limited Signals*, The Bell System Technical Journal, July 1962.

[19] H.J. Landau, H. Widom, *Eigenvalue Distribution of Time and Frequency Limiting*, Journal of Mathematical Analysis and Applications, 77, 469-481 (1980).

[20] D. Slepian, H.O. Pollak, *Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty - I*, The Bell System Technical Journal, January 1961.

[21] D. Slepian, *Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty - IV: Extensions to Many Dimensions, Generalized Prolate Spheroidal Wave Functions*, The Bell System Technical Journal, November 1964.

[22] D. Slepian, *Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty - V: The Discrete Case*, The Bell System Technical Journal, May-June 1978.

[23] D. Slepian, *Some Comments on Fourier Analysis, Uncertainty, and Modeling* SIAM Review, V. 25, No. 3, July 1983.

[24] J. Stoer and R. Bulirsch, *Introduction To Numerical Analysis*, 2'nd ed., Springer-Verlag, 1992

[25] N. Yarvin and V. Rokhlin, *Generalized Gaussian Quadratures and Singular Value Decompositions of Integral Operators*, SIAM Journal of Scientific Computing, Vol. 20, No. 2, pp. 699-718 (1998).

Table 1: Quadrature performance for varying band limits, for $\varepsilon = 10^{-7}$

c	n	Maximum Errors		N_{pol}
		Roots	Refined	
10.0	9	0.96E-05	0.51E-07	13
20.0	13	0.17E-04	0.94E-07	19
30.0	17	0.12E-04	0.50E-07	25
40.0	20	0.70E-05	0.30E-06	31
50.0	24	0.35E-05	0.83E-07	37
60.0	27	0.25E-04	0.27E-06	43
70.0	31	0.11E-04	0.66E-07	48
80.0	34	0.48E-05	0.17E-06	54
90.0	38	0.21E-05	0.40E-07	59
100.0	41	0.12E-04	0.91E-07	65
200.0	74	0.24E-05	0.86E-07	118
300.0	106	0.32E-05	0.21E-06	171
400.0	139	0.52E-05	0.62E-07	223
500.0	171	0.56E-05	0.88E-07	275
600.0	203	0.58E-05	0.11E-06	326
700.0	235	0.57E-05	0.12E-06	377
800.0	267	0.55E-05	0.13E-06	428
900.0	299	0.53E-05	0.14E-06	479
1000.0	331	0.50E-05	0.14E-06	530
1200.0	395	0.44E-05	0.13E-06	632
1400.0	459	0.38E-05	0.11E-06	734
1600.0	523	0.31E-05	0.97E-07	835
1800.0	587	0.28E-05	0.80E-07	937
2000.0	651	0.23E-05	0.64E-07	1038
2400.0	778	0.29E-05	0.15E-06	1240
2800.0	906	0.19E-05	0.84E-07	1442
4000.0	1288	0.37E-05	0.17E-06	2047

Table 2: Quadrature performance for varying precisions, for $c = 50$

ε	n	Maximum Errors		N_{pol}
		Roots	Refined	
0.10E-01	19	0.45E-01	0.10E-01	30
0.10E-02	20	0.70E-02	0.13E-02	32
0.10E-03	21	0.91E-03	0.14E-03	33
0.10E-04	22	0.82E-04	0.13E-04	34
0.10E-05	23	0.54E-04	0.11E-05	36
0.10E-06	24	0.35E-05	0.83E-07	37
0.10E-07	25	0.33E-05	0.57E-08	38
0.10E-08	26	0.18E-06	0.36E-09	39
0.10E-09	26	0.18E-06	0.36E-09	40
0.10E-10	27	0.17E-06	0.21E-10	42
0.10E-11	28	0.79E-08	0.11E-11	43
0.10E-12	29	0.78E-08	0.56E-13	45
0.10E-13	30	0.31E-09	0.27E-14	55

Table 3: Interpolation performance for varying band limits, for $\varepsilon = 10^{-7}$

c	n	Maximum Errors		N_{pol}	
		Roots	Refined	Cheb.	Leg.
5.0	13	0.12E-06	0.12E-06	17	17
10.0	18	0.12E-06	0.13E-06	24	25
15.0	22	0.24E-06	0.25E-06	31	32
20.0	26	0.26E-06	0.28E-06	37	39
25.0	30	0.22E-06	0.23E-06	43	45
30.0	33	0.67E-06	0.73E-06	49	51
35.0	37	0.42E-06	0.46E-06	55	57
40.0	41	0.25E-06	0.27E-06	61	63
45.0	44	0.54E-06	0.60E-06	67	69
50.0	48	0.29E-06	0.33E-06	73	75
100.0	82	0.39E-06	0.46E-06	128	131
150.0	115	0.52E-06	0.64E-06	182	186
200.0	147	0.12E-05	0.15E-05	235	239
250.0	180	0.83E-06	0.11E-05	287	292
300.0	212	0.13E-05	0.17E-05	340	345
350.0	245	0.75E-06	0.10E-05	392	398
400.0	277	0.10E-05	0.14E-05	443	450
450.0	309	0.13E-05	0.18E-05	495	502
500.0	341	0.16E-05	0.22E-05	547	554
1000.0	662	0.16E-05	0.24E-05	1058	1068
1500.0	982	0.15E-05	0.25E-05	1566	1578
2000.0	1301	0.20E-05	0.35E-05	2072	2086

Table 4: Interpolation performance for varying precisions, for $c = 25$

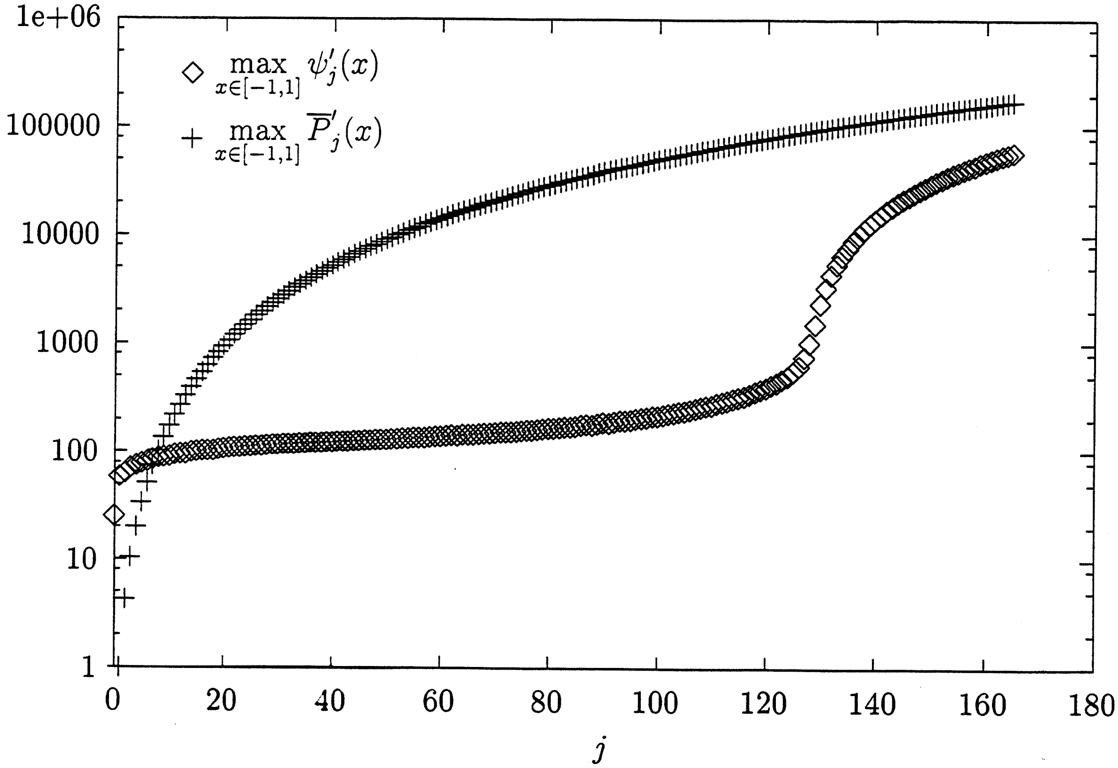
ε	n	Maximum Errors		N_{pol}	
		Roots	Refined	Cheb.	Leg.
0.10E-01	21	0.38E-01	0.43E-01	31	34
0.10E-02	23	0.37E-02	0.41E-02	34	36
0.10E-03	25	0.29E-03	0.31E-03	37	39
0.10E-04	26	0.74E-04	0.81E-04	39	41
0.10E-05	28	0.44E-05	0.47E-05	41	43
0.10E-06	30	0.22E-06	0.23E-06	43	45
0.10E-07	31	0.46E-07	0.49E-07	45	47
0.10E-08	32	0.95E-08	0.10E-07	47	49
0.10E-09	34	0.36E-09	0.38E-09	49	51
0.10E-10	35	0.67E-10	0.70E-10	51	52
0.10E-11	37	0.21E-11	0.22E-11	53	54
0.10E-12	38	0.36E-12	0.37E-12	54	56
0.10E-13	39	0.59E-13	0.63E-13	98	61

Table 5: Quadrature nodes for band-limited functions, with $c = 50$ and $\varepsilon = 10^{-7}$

This table contains only half of the nodes and weights, in particular those for which the node is less than or equal to zero; reflecting these nodes around zero yields the remaining nodes, the weight for the node at $-x$ being the same as the weight for the node at x.

Node	Weight
−.9904522459960804E+00	0.2413064234922188E−01
−.9525601106643832E+00	0.5024347217095568E−01
−.8927960861459153E+00	0.6801787677830858E−01
−.8186117530609125E+00	0.7952155999100788E−01
−.7350624131965875E+00	0.8706680708376023E−01
−.6452878027260844E+00	0.9216240765763570E−01
−.5512554698695428E+00	0.9569254015486106E−01
−.4542505281525226E+00	0.9817257766311556E−01
−.3551568458127944E+00	0.9990914516102242E−01
−.2546173463813596E+00	0.1010880172648715E+00
−.1531287781860989E+00	0.1018214308931439E+00
−.5110121484050418E−01	0.1021735189986602E+00

Figure 2: Maximum norms of derivatives of prolate spheroidal wave functions for $c = 200$, and of normalized Legendre polynomials



Norms of eigenvalues λ_j for $c = 200$:

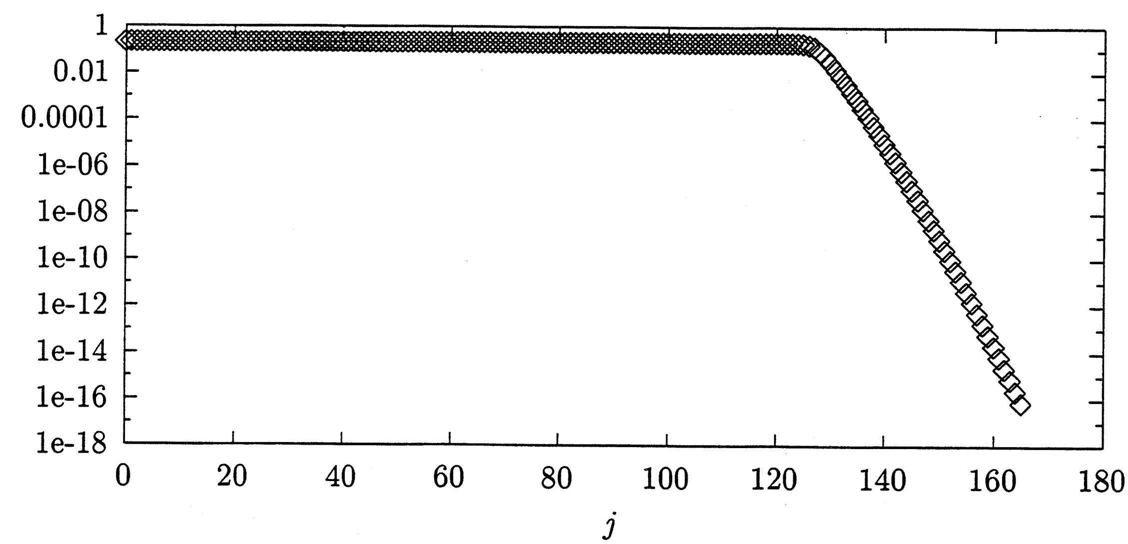


Table 6: Quadrature nodes for band-limited functions, with $c = 150$ and $\varepsilon = 10^{-14}$
This table contains only half of the nodes and weights, in particular those for which the node is less than or equal to zero; reflecting these nodes around zero yields the remaining nodes, the weight for the node at $-x$ being the same as the weight for the node at x.

Node	Weight
−.9982883010959975E+00	0.4374483371752129E−02
−.9911354691596528E+00	0.9842619236149078E−02
−.9788315280982487E+00	0.1463518300250369E−01
−.9621348937901911E+00	0.1862396111287527E−01
−.9418386698454396E+00	0.2184988739217138E−01
−.9186509576802944E+00	0.2442858670932862E−01
−.8931541850293142E+00	0.2648864579258096E−01
−.8658083894041821E+00	0.2814375940413615E−01
−.8369709588254746E+00	0.2948528624795690E−01
−.8069187108185302E+00	0.3058356160435090E−01
−.7758670331396409E+00	0.3149181066633766E−01
−.7439849501152674E+00	0.3225015506203403E−01
−.7114064976175457E+00	0.3288893713079314E−01
−.6782391686910609E+00	0.3343126421620424E−01
−.6445701594098660E+00	0.3389488931551181E−01
−.6104710013384929E+00	0.3429358206877410E−01
−.5760010202980960E+00	0.3463812513892117E−01
−.5412099413257457E+00	0.3493704033879884E−01
−.5061398697742787E+00	0.3519712095895683E−01
−.4708268134473433E+00	0.3542382499917732E−01
−.4353018643598344E+00	0.3562156808557525E−01
−.3995921259242572E+00	0.3579394352776868E−01
−.3637214481257228E+00	0.3594388900778062E−01
−.3277110167114320E+00	0.3607381381247460E−01
−.2915798305819667E+00	0.3618569660385742E−01
−.2553450930388687E+00	0.3628116095737887E−01
−.2190225363501577E+00	0.3636153393399723E−01
−.1826266945721476E+00	0.3642789154364812E−01
−.1461711362450572E+00	0.3648109393796617E−01
−.1096686661347072E+00	0.3652181242257066E−01
−.7313150339365902E−01	0.3655054982303338E−01
−.3657144220122915E−01	0.3656765531685031E−01
0	0.3657333451556860E−01